On optimal polynomial meshes

Let $P_n^d$ be the space of real algebraic polynomials of $d$ variables and degree at most $n$, $K \subset \mathbb{R}^d$ a compact set, $||p||_K := \sup_{x \in K} |p(x)|$ the usual supremum norm on $K$, $\text{card}(Y)$ the cardinality of a finite set $Y$. A family of sets $Y = \{Y_n \subset K, n \in \mathbb{N}\}$ is called an admissible mesh in $K$ if there exists a constant $c_1 > 0$ depending only on $K$ such that

$$||p||_K \leq c_1 ||p||_{Y_n}, \quad p \in P_n^d, n \in \mathbb{N},$$

where the cardinality of $Y_n$ grows at most polynomially. If $\text{card}(Y_n) \leq c_2 n^d, n \in \mathbb{N}$ with some $c_2 > 0$ depending only on $K$ then we say that the admissible mesh is optimal. This notion of admissible meshes is related to norming sets which are widely used in the literature. In this talk we shall present some general families of sets possessing admissible meshes which are optimal or near optimal in the sense that the cardinality of sets $Y_n$ does not grow too fast. In particular, we shall see that graph domains bounded by polynomial graphs, convex polytopes and star like sets with $C^2$ boundary possess optimal admissible meshes. In addition, graph domains with piecewise analytic boundary and any convex sets in $\mathbb{R}^2$ possess almost optimal admissible meshes in the sense that the cardinality of admissible meshes is larger than optimal only by a $\log n$ factor.