Coordinates and Change of basis

I will get back to the example done in class on April 2 to use notation in accordance to the text.

Note: In class I was using the coordinates of the vector in a basis as a row vector and the text uses them as a column vector. This changes a bit the way we write the matrices and I do not want to confuse you by having a different notation than the text.

So let us start with a simple example in R^2. Consider two bases, $B=\{(1,0),(0,1)\}$, the standard basis, and $B'=\{(1,1),(1,2)\}$. We also consider a vector $x=(2,3)$ whose coordinates in basis B are $[x]_B=\begin{bmatrix}2\\3\end{bmatrix}=[2,3]^T$ meaning that $x=2(1,0)+3(0,1)$. Now, we want to determine the coordinates of x in the new basis $B'=\{(1,1),(1,2)\}$. As we noticed in class, we can write $x=(1,1)+(1,2)$, so $[x]_{B'}=\begin{bmatrix}1\\1\end{bmatrix}=[1,1]^T$.

We will try to develop a general method for such transformations.

Since B is a basis we may represent the vectors in the new basis B' in a unique way as linear combinations of the vectors in B.

$$\begin{aligned}(1,1)&=(1,0)+(0,1)\\(1,2)&=(1,0)+2(0,1)\end{aligned}\qquad(1)$$

Denote by $P=\begin{bmatrix}1&1\\1&2\end{bmatrix}$ the transpose of the coefficient matrix above.

Note: In this particular example it makes no difference since the matrix is symmetric, but you will see in the general case.

If we denote in the system (1) above: $v_1=(1,1), v_2=(1,2), u_1=(1,0), u_2=(0,1)$ we can write:

$$\begin{bmatrix}v_1 & v_2\end{bmatrix}=\begin{bmatrix}u_1 & u_2\end{bmatrix}P \qquad (2)$$

Then, since the matrix P is invertible, we can solve in (2) to obtain the vectors of the old basis B in terms of the vectors of B', by writing: $\begin{bmatrix}u_1 & u_2\end{bmatrix}=\begin{bmatrix}v_1 & v_2\end{bmatrix}P^{-1}$.

Now, take the vector $x=(2,3); [x]_B=\begin{bmatrix}2\\3\end{bmatrix}=[2,3]^T$. Then we can write

$$x=\begin{bmatrix}u_1 & u_2\end{bmatrix}\begin{bmatrix}2\\3\end{bmatrix}=\begin{bmatrix}v_1 & v_2\end{bmatrix}P^{-1}\begin{bmatrix}2\\3\end{bmatrix} \quad (3)$$

We can compute $P^{-1}=\begin{bmatrix}2&-1\\-1&1\end{bmatrix}$, and (3) becomes

$$x=\begin{bmatrix}v_1 & v_2\end{bmatrix}\begin{bmatrix}2&-1\\-1&1\end{bmatrix}\begin{bmatrix}2\\3\end{bmatrix}=\begin{bmatrix}v_1 & v_2\end{bmatrix}\begin{bmatrix}1\\1\end{bmatrix}=v_1+v_2,$$

which means that $[x]_{B'}=\begin{bmatrix}1\\1\end{bmatrix}=[1,1]^T$, as we had above.

We now look at the general case, which is considered on page 246 in your text. We denote as before by $B = \{u_1, u_2, ..., u_n\}$ the old basis, and $B' = \{v_1, v_2, ..., v_n\}$ the new basis. We can write

$$\begin{aligned} v_1 &= c_{11}u_1 + c_{21}u_2 + ... + c_{n1}u_n \\ v_2 &= c_{12}u_1 + c_{22}u_2 + ... + c_{n2}u_n \\ &\vdots \\ v_n &= c_{1n}u_1 + c_{2n}u_2 + ... + c_{nn}u_n \end{aligned} \qquad (4)$$

Now, if we denote $P = \begin{bmatrix} c_{11} & c_{12} & ... & c_{1n} \\ c_{21} & c_{22} & ... & c_{2n} \\ \vdots & \vdots & ... & \vdots \\ c_{n1} & c_{n2} & ... & c_{nn} \end{bmatrix}$, the system (4) above can be written in matrix form: $\begin{bmatrix} v_1 & v_2 & ... & v_n \end{bmatrix} = \begin{bmatrix} u_1 & u_2 & ... & u_n \end{bmatrix} P$ (5).

Since P must be invertible (because the new set of vectors forms a basis), we can solve in (5) to obtain the vectors of the old basis B in terms of the vectors of B', by writing: $\begin{bmatrix} u_1 & u_2 & ... & u_n \end{bmatrix} = \begin{bmatrix} v_1 & v_2 & ... & v_n \end{bmatrix} P^{-1}$ (6).

Suppose now we have a vector $x = a_1u_1 + a_2u_2 + ... + a_nu_n = \begin{bmatrix} u_1 & u_2 & ... & u_n \end{bmatrix} \begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{bmatrix}$, so the coordinates of x in basis B are $[x]_B = \begin{bmatrix} a_1 & a_2 & ... & a_n \end{bmatrix}^T$.

Using equation (6) in the above expression we get:

$$x = \begin{bmatrix} u_1 & u_2 & ... & u_n \end{bmatrix} \begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{bmatrix} = \begin{bmatrix} v_1 & v_2 & ... & v_n \end{bmatrix} P^{-1} \begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{bmatrix} \text{ (7).}$$

From (7) it follows that the coordinates of vector x in the basis B' are given by

$$[x]_{B'} = P^{-1} \begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{bmatrix} = P^{-1}[x]_B. \quad (8)$$

Observation: The matrix P^{-1} in all the above computations is the matrix Q in your text.

We call the **transition matrix** from basis B to basis B', the matrix Q whose columns are the coordinates of the basis vectors in B expressed with respect to the basis B'.

$$Q = \begin{bmatrix} [u_1]_{B'} & [u_2]_{B'} & ... & [u_n]_{B'} \end{bmatrix}$$

The inverse of this matrix P, gives the transition from B' to B, and is given by:

$$P = \left[[v_1]_B \quad [v_2]_B \quad \dots \quad [v_n]_B \right]$$

The **change of coordinate** formula (8) is the expressed as:

$$[x]_{B'} = Q[x]_B, \text{ or } [x]_B = P[x]_{B'}$$

To find the transition matrix from an **old basis** B to a **new basis** B' do the following:

Step 1. Form the matrix $\left[B' \mid B \right]$ in which the vectors of the bases are arranged as columns.

Step 2. Use elementary row operations to reduce the matrix in Step 1 to its reduced row echelon form.

Step 3. The resulting matrix will be $\left[I \mid Q \right]$, Q being the transition matrix.

Example 1. Let us work **Exercise 2** in section 4.7. We have: $B = \{(-1,4),(4,-1)\}, [x]_B = [-2,3]^T$. We want to find the coordinates of x in $B' = \{(1,0),(0,1)\}$.

To find the transition matrix we write:

$$\left[B' \mid B \right] = \begin{bmatrix} 1 & 0 & -1 & 4 \\ 0 & 1 & 4 & -1 \end{bmatrix} = \left[I \mid Q \right].$$

So, $[x]_{B'} = Q[x]_B = \begin{bmatrix} -1 & 4 \\ 4 & -1 \end{bmatrix} \begin{bmatrix} -2 \\ 3 \end{bmatrix} = \begin{bmatrix} 14 \\ -11 \end{bmatrix}$.

We may check directly that $x = -2(-1,4) + 3(4,-1) = (14,-11) = 14(1,0) - 11(0,1)$.

Exercise 6. This is just like exercise 2 above. Since the new basis is the standard basis in R^4, the transition matrix is just

$Q = \begin{bmatrix} 4 & 0 & -3 & 0 \\ 0 & 5 & 4 & 1 \\ 7 & -1 & 2 & 5 \\ 3 & -1 & 1 & 0 \end{bmatrix}$ and we have $[x]_{B'} = \begin{bmatrix} 4 & 0 & -3 & 0 \\ 0 & 5 & 4 & 1 \\ 7 & -1 & 2 & 5 \\ 3 & -1 & 1 & 0 \end{bmatrix} \begin{bmatrix} -2 \\ 3 \\ 4 \\ 1 \end{bmatrix} = \begin{bmatrix} -20 \\ 32 \\ -4 \\ -5 \end{bmatrix}$.

Exercise 10. The given vector $x = (3, -\frac{1}{2}, 8) = 3(1,0,0) - \frac{1}{2}(0,1,0) + 8(0,0,1)$ is with respect to the standard basis in R^3. Here the new basis is the basis *B*. We need to find $[x]_B = Q[x]_{\text{standard}}$. We proceed to finding Q, the transition matrix from the standard basis to the basis *B*. We have, with Maple's valuable help:

$$\begin{bmatrix} \frac{3}{2} & \frac{3}{4} & 1 & 1 & 0 & 0 \\ 4 & \frac{5}{2} & \frac{1}{2} & 0 & 1 & 0 \\ 1 & 0 & 2 & 0 & 0 & 1 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 0 & -8 & \frac{12}{5} & \frac{17}{5} \\ 0 & 1 & 0 & 12 & \frac{-16}{5} & \frac{-26}{5} \\ 0 & 0 & 1 & 4 & \frac{-6}{5} & \frac{-6}{5} \end{bmatrix}$$, therefore

$$[x]_B = \begin{bmatrix} -8 & \frac{12}{5} & \frac{17}{5} \\ 12 & \frac{-16}{5} & \frac{-26}{5} \\ 4 & \frac{-6}{5} & \frac{-6}{5} \end{bmatrix} \begin{bmatrix} 3 \\ \frac{-1}{2} \\ 8 \end{bmatrix} = \begin{bmatrix} 2 \\ -4 \\ 3 \end{bmatrix}.$$

Now, if you've made it up to here, you may work some problems on your own.
Bring them on Monday to class to get some extra credit!!
I would like to see the work done, not just the answer, please.
Exercises: 4, 9, 11, 12 (I strongly recommend using Maple or similar software), 18, 20.