Homework #5 solutions

Ex. 23 section 4.2. We are looking at the set $S = \left\{ A = \begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix}, a, b \in R \right\}$. The set is closed under the operation of addition of matrices and multiplication by a scalar, since the sum of two diagonal matrices is a diagonal matrix, and if we multiply a diagonal matrix by a scalar we get a diagonal matrix. All the remaining axioms of vector space will be satisfied as they are satisfied by the addition and multiplication of 2×2 matrices. In particular, the additive identity is the zero matrix $\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$, and the additive inverse of a diagonal matrix $A = \begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix}$ is $-A = \begin{bmatrix} -a & 0 \\ 0 & -b \end{bmatrix}$.

Ex. 25 (b) section 4.2. R^2 is not a vector space with the operations defined since it fails to have an additive identity. There does not exist $(a,b) \in R^2 : (a,b) + (x,y) = (x,y)$ since no matter what the values of b and $y \neq 0$ $(0,b) + (x,y) = (x,0) \neq (x,y)$.
Or, a different argument: if it were a vector space then
$2\cdot(x,y) = (1+1)\cdot(x,y) = (x,y) + (x,y) = (2x,0)$. But $2(x,y) = (2x,2y)$, so the equality will not hold if $(x,y) = (2,3)$ since $2(2,3) = (4,6) \neq (2,3) + (2,3) = (4,0)$.

Ex. 28 section 4.2. We will check the 10 axioms. Let $S = \{(x,2x) : x \in R\}$.
(1) If $(x_1,2x_1),(x_2,2x_2) \in S : (x_1,2x_1) + (x_2,2x_2) = (x_1+x_2, 2(x_1+x_2)) \in S$
(2) $(x_1,2x_1) + (x_2,2x_2) = (x_1+x_2, 2(x_1+x_2)) = (x_2,2x_2) + (x_1,2x_1)$
(3) $(x_1,2x_1) + (x_2,2x_2)) + (x_3,2x_3) = (x_1+x_2+x_3, 2(x_1+x_2+x_3)) = (x_1,2x_1) + ((x_2,2x_2) + (x_3,2x_3))$

(4) $(0,0) \in S$ and $(0,0) + (x,2x) = (x,2x)$ so we have an additive identity.
(5) $(x,2x) + (-x,-2x) = (0,0)$.
(6) For $c \in R : c(x,2x) = (cx,2cx) \in S$ so the set is closed under the scalar multiplication.
(7) Let $k \in R, (x_1,2x_1),(x_2,2x_2) \in S$. Then

$$k((x_1,2x_1) + (x_2,2x_2)) = (k(x_1+x_2), k(2x_1+2x_2)) = k(x_1,2x_1) + k(x_2,2x_2)$$

(8) Let $k,l \in R, (x,2x) \in S$. Then

$$(k+l)(x,2x) = ((k+l)x, 2(k+l)x) = (kx,2kx) + (lx,2lx) = k(x,2x) + l(x,2x).$$

(9) Let $k,l \in R, (x,2x) \in S$. Then $k\cdot(l\cdot(x,2x)) = k(lx,2lx) = (klx,2klx) = (kl)\cdot(x,2x)$.
(10) And finally $1\cdot(x,2x) = (1\cdot x, 2\cdot 1\cdot x) = (x,2x)$
So all the axioms are satisfied and therefore S is a vector space.

Ex. 29 section 4.2. Axioms (1) and (6) are satisfied and there is no problem in checking that the "additive" operation is commutative and associative. Let us find the additive identity.
We have $(a,b) + (x,y) = (ax,by) = (x,y)$, and therefore $a = b = 1$.

Now for the additive inverse: let $(x, y) \in R^2$. We try to find

$$(x', y') : (x, y) + (x', y') = (xx', yy') = (1,1)$$

Then $x' = x^{-1}, y' = y^{-1}$. But notice that if $(x, y) = (1, 0)$ there does not exist an additive inverse. Therefore the set is not a vector space with the defined operations.

Ex. 2 section 4.3. $W = \{(x, y, 2x - 3y) : x, y \in R\}$. We will check that the set is closed under addition and scalar multiplication.
Let $(x, y, 2x - 3y), (x_1, y_1, 2x_1 - 3y_1) \in W$. Then

$$(x, y, 2x - 3y) + (x_1, y_1, 2x_1 - 3y_1) = (x + x_1, y + y_1, 2(x + x_1) - 3(y + y_1)) \in W$$

Now let $c \in R$. Then $c(x, y, 2x - 3y) = (cx, cy, 2cx - 3cy) \in W$ and we proved that W is a subspace of R^3.

Ex. 3 section 4.3. Let $A = \begin{bmatrix} 0 & a \\ b & 0 \end{bmatrix}, A_1 = \begin{bmatrix} 0 & a_1 \\ b_1 & 0 \end{bmatrix}$. Then $A + A_1 = \begin{bmatrix} 0 & a + a_1 \\ b + b_1 & 0 \end{bmatrix} \in W$.
Also, if $c \in R, cA = \begin{bmatrix} 0 & ca \\ cb & 0 \end{bmatrix} \in W$, so the axioms for subspace check.

Ex. 13(e) section 4.3. We will prove that the given set is a subspace of $C(-\infty, \infty)$.
Denote $W = \{ f \in C(-\infty, \infty) : f(0) = 0\}$. Let $f, g \in W, c \in R$. We have $(f + g)(0) = f(0) + g(0) = 0,$ and $(cf)(0) = cf(0) = 0$. Therefore $f + g \in W$ and $cf \in W$, proving that W is a subspace of $C(-\infty, \infty)$.

Ex. 16 section 4.3. The set is not a subspace because it is not closed under any of the operations. We show that on a particular example. Let $(1,1,4) \in W$. Then $0(1,1,4) = (0,0,0) \notin W$.

Ex. 17 section 4.3 This is very similar to exercise 2. The set is closed under both operations since $(x, y, x + 2y) + (x_1, y_1, x_1 + 2y_1) = (x + x_1, y + y_1, (x + x_1) + 2(y + y_1)) \in W$, and $c(x, y, x + 2y) = (cx, cy, cx + 2cy) \in W$.

Ex. 26 section 4.3. $W = \{x \in R^n : Ax = 0\}$. Let $x, x_1 \in R^n : Ax = 0, Ax_1 = 0$. Then $A(x + x_1) = Ax + Ax_1 = 0$, and $A(cx) = cAx = 0$ proving that the set is closed under both operations, and therefore it is a subspace.