Spring 2008 Math 3260

Practice test #3 Answer key

1. In order to check that a transformation is linear you have to check the two properties, $T((x_1, y_1) + (x_2, y_2)) = T(x_1, y_1) + T(x_2, y_2)$ and $T(c(x, y)) = cT(x, y)$.

Since the transformation is not linear, it is sufficient to show that one of the properties fails to be satisfied for a particular choice of values of *x*, *y* and/or *c*.

In particular, if *x*=*y*=0, and *c*=2, the second property says that $T(2(0,0)) = 2T(0,0)$. But $T(2(0,0) = T(0,0) = 5,$ whereas $2T(0,0) = 10$, proving that the transformation is not linear.

2. The transformation is not linear. Indeed, if we consider $T(2(x, y)) = T((2x, 2y)) = (2x, 1)$, whereas $2T((x, y)) = 2(x, 1) = (2x, 2) \neq T(2(x, y))$.

3. (a) It is pretty obvious what the standard matrix should be, but to follow the directions of the problem we will find the values of T applied to the standard basis vectors. We have $T(1,0) = (1,0),$ and $T(0,1) = (0,-1),$ and therefore $T = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}.$

(b) Notice that $T : R^4 \to R,$ and as before we get $T = \begin{bmatrix} 4 & 6 & 7 & 8 \end{bmatrix}.$

(c) $T : R^2 \to R^3$. We need to use the vectors of the basis now. Let $e_1 = (1,0),$ and $e_2 = (0,1).$ We have $(-1,1) = -e_1 + e_2,$ and $(2,-1) = 2e_1 - e_2.$ Applying the linearity of the transformation, we have:

$$\begin{cases} T(-1,1) = -T(e_1) + T(e_2) = (3,0,-1) \\ T(2,-1) = 2T(e_1) - T(e_2) = (0,-2,4). \end{cases}$$

We may now solve the system by adding the two equations to obtain $T(e_1) = (3,-2,3),$ and then by substituting, $T(e_2) = (6,-2,2)$.

Therefore, $T = \begin{bmatrix} 3 & 6 \\ -2 & -2 \\ 3 & 2 \end{bmatrix}.$

4. We have $T = \begin{bmatrix} 3 & 3 \\ -1 & -1 \end{bmatrix}$, and since det(T)=0 it follows that T is not one-to-one. Since det(T)=0, it also follows that the range of *T* cannot be R^2 In order to find a vector that is not in the range, notice that the vectors **w** in the range of *T* have the property that $w_1 = -3w_2$. Therefore, if we take the vector **w**=(0,1), there is not vector (*x*,*y*) such that $T(x, y) = (0,1).$

5. The equation of the line through (4,1) and the origin is $x - 4y = 0$, so a possible matrix is $A = \begin{bmatrix} 1 & -4 \\ 2 & -8 \end{bmatrix}.$

6. We begin by checking whether the vectors are linearly independent. Since $\dim(P_3) = 4$, if we prove linear independence automatically the vectors will span the

space. Let $c_1(t^3-2t^2+1)+c_2(t^2-4)+c_3(t^3+2t)+c_4 5t=0$. Then if we group on power s of *t*, all coefficients must be zero. We get:

$$\begin{cases} c_1+c_3=0 \\ -2c_1+c_2=0 \\ 2c_3+c_4=0 \\ c_1-4c_2=0 \end{cases}$$

The matrix of the above system has a non-zero determinant, and therefore the system has only the trivial solution. Therefore, $c_1=c_2=c_3=c_4=0$ and the vectors are linearly independent. Since the dimension of the space is 4 and we have 4 linearly independent vectors, they form a basis.

7. It can be checked that the set is a subspace of R^3. If we denote $x=2s-t, y=s, z=t$ we can see that W is the set of points satisfying $x=2y-z,$ or $x-2y+z=0$, which is a plane going through the origin. To find a basis in this subspace we write:

$$(2s-t,s,t)=s(2,1,0)+t(-1,0,1)$$

Then the vectors $(2,1,0),(-1,0,1)$ span W. Moreover, they are linearly independent since

$$s(2,1,0)+t(-1,0,1)=(0,0,0)\Rightarrow s=t=0$$

Therefore they are a basis in W.

8. To find a basis in R^2 that includes (1,1) it is enough to find a vector which is not a multiple of (1,1). Any such vector will do. Take for example (2,1). Then we have two linearly independent vectors, $(1,1),(2,1)$ and since $\dim(R^2)=2$, the two vectors are a basis.

9. We will use the linearity of the operator to determine the value of T on the standard basis vectors, $T(1,0)$, and $T(0,1)$. Refer to the standard basis vectors as e_1,e_2. We have:

$$T(1,2)=T(e_1+2e_2)=T(e_1)+2T(e_2)=(-2,3)$$
$$T(1,-1)=T(e_1-e_2)=T(e_1)-T(e_2)=(5,2)$$

We can solve the above system to obtain: $3T(e_2)=(-7,1), 3T(e_1)=(8,7)$. Therefore $T=\frac{1}{3}\begin{bmatrix} 8 & -7 \\ 7 & 1 \end{bmatrix}$, and $T(7,5)=\frac{1}{3}\begin{bmatrix} 8 & -7 \\ 7 & 1 \end{bmatrix}\begin{bmatrix} 7 \\ 5 \end{bmatrix}=\begin{bmatrix} 7 \\ 18 \end{bmatrix}$.

10. a) Let $u_1=(x_1,y_1),u_2=(x_2,y_2)\in R^2, c_1,c_2\in R$. We have

$$T(c_1u_1+c_2u_2)=(c_1x_1+c_2x_2+c_1y_1+c_2y_2, c_1x_1+c_2x_2-c_1y_1-c_2y_2, 3(c_1x_1+c_2x_2)+2(c_1y_1+c_2y_2))$$
$$=c_1(x_1+y_1,x_1-y_1,3x_1+2y_1)+c_2(x_2+y_2,x_2-y_2,3x_2+2y_2)=c_1T(u_1)+c_2T(u_2)$$

so the transformation is linear.

b) In order to decide whether *T* is one-to-one we need to see whether $T(x,y)=(0,0,0)$ has only the trivial solution $(x,y)=(0,0)$. We write the matrix of the transformation and reduce it to its reduced row echelon form:

$T=\begin{bmatrix}1 & 1\\ 1 & -1\\ 3 & 2\end{bmatrix}\sim\begin{bmatrix}1 & 0\\ 0 & 1\\ 0 & 0\end{bmatrix}$. Since the matrix has two leading ones, and we solve for two unknowns, the system will have only the trivial solution, and therefore the transformation is one-to-one.

c) Let now $(w_1, w_2, w_3)\in R^3$. We need to see whether there exists $(x,y)\in R^2$ such that $T(x,y)=(w_1,w_2,w_3)$. We write the augmented matrix of the system and reduce it:

$$\begin{bmatrix}1 & 1 & w_1\\ 1 & -1 & w_2\\ 3 & 2 & w_3\end{bmatrix}\sim\begin{bmatrix}1 & 1 & w_1\\ 0 & -2 & w_2-w_1\\ 0 & -1 & w_3-3w_1\end{bmatrix}\sim\begin{bmatrix}1 & 1 & w_1\\ 0 & 1 & -\frac{1}{2}(w_2-w_1)\\ 0 & 0 & w_3-3w_1-\frac{1}{2}(w_2-w_1)\end{bmatrix}$$

The system will be consistent if and only if $w_3-3w_1-\frac{1}{2}(w_2-w_1)=0, 2w_3-5w_1-w_2=0$.

A vector that is not in the range(T) is $(1,0,1)$.

11. Of course, since one vector is a multiple of the other the two vectors cannot span R^2. We will actually find a vector in R^2 that is not in $span\{(1,1),(-2,-2)\}$. Consider the vector (1,0). If it were in the subspace spanned by the two given vectors, then there must exist c_1, c_2 such that $c_1(1,1)+c_2(-2,-2)=(1,0)$. But if we write the system corresponding to this vector equation we obtain $c_1-2c_2=1=0$. Therefore there do not exist such scalars, and thus (1,0) is not in $span\{(1,1),(-2,-2)\}$.