

### A final set of problems

... but if you enjoyed Linear Algebra, you may ask for more and I can provide.

1. Determine the values of  $k$  such that the following system of equations does not have a unique solution:

$$\begin{cases} x + y + kz = 3 \\ x + ky + z = 2 \\ kx + y + z = 1 \end{cases}$$

Equivalently, you can ask the question in the language of the vector spaces associated with the system matrix, call it  $A$ . So, as you can see, this is a non-homogeneous system, and as we discussed, the solution can be written as  $(x_0, y_0, z_0) + (x_1, y_1, z_1)$ , where  $(x_0, y_0, z_0)$  is the general solution to the homogeneous linear system, and  $(x_1, y_1, z_1)$  is a particular solution to the non-homogeneous system. So, the system not having a unique solution can mean either infinitely many solutions (i.e.  $\dim(\text{nullspace}(A)) > 0$ ), or no solution at all (i.e.  $(3, 2, 1)$  is in the column space of the matrix).

2. a) Find a linear system of two equations in two unknowns  $x$ , and  $y$ , that has the solution set given by the parametric representation  $x = t, y = 3t - 4$ , where  $t$  is any real number.  
b) Geometrically, what do the two parametric equations represent? Graph the solution set from above.
3. Find values of  $a, b$ , and  $c$  such that the following system of equations has (a) exactly one solution, (b) an infinite set of solutions, and (c) no solution.

$$\begin{cases} x + 5y + z = 0 \\ x + 6y - z = 0 \\ 2x + ay + bz = c \end{cases}$$

4. Determine a polynomial of degree 2 that passes through the points  $(2, 5), (3, 2), (4, 5)$ .
5. A square matrix is called **skew-symmetric** if  $A^T = -A$ .  
a) Determine whether the following matrices are symmetric, skew-symmetric or neither:

$$A = \begin{bmatrix} 0 & 2 \\ -2 & 0 \end{bmatrix}, B = \begin{bmatrix} 2 & 1 \\ 1 & 3 \end{bmatrix}, C = \begin{bmatrix} 0 & 2 & -1 \\ -2 & 0 & -3 \\ 1 & 3 & 0 \end{bmatrix}$$

- b) Prove that the main diagonal of a skew-symmetric matrix consists entirely of zeros. Do not consider particular matrices, such as of order 2 or 3.
- c) What can you say about the product of two skew-symmetric matrices: is it skew-symmetric, symmetric or neither? Assume that the product exists.

6. Does there exist a matrix  $A$  such that

$$A \begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} b & d \\ a & c \end{bmatrix}, \text{ for all values of } a, b, c \text{ and } d?$$

7. Show that if a square matrix satisfies  $A^2 - 5A + 6I = 0$ , then so does  $A^T$ .

8. Show that if  $A$  is invertible, then  $A + B$  and  $I + BA^{-1}$  are both invertible or both not invertible.

9. For which values of  $k$  does  $A$  fail to be invertible?

$$A = \begin{bmatrix} 1 & 2 & 4 \\ 3 & 1 & 6 \\ k & 3 & 2 \end{bmatrix}.$$

10. Without directly evaluating, show that  $\det \begin{bmatrix} b+c & c+a & b+a \\ a & b & c \\ 1 & 1 & 1 \end{bmatrix} = 0$ .

11. Show that if  $T: R^n \rightarrow R^m$  is a linear transformation, then  $T(0_{R^n}) = 0_{R^m}$ .

12. Let  $T: R^3 \rightarrow R^3$  be multiplication by the matrix  $T = \begin{bmatrix} -1 & 3 & 0 \\ 2 & 1 & 2 \\ 4 & 5 & -3 \end{bmatrix}$ , and let

$e_1, e_2$ , and  $e_3$  be the standard basis vectors in  $R^3$ . Find the following:

a)  $T(e_1), T(e_2), T(e_3)$ .

b)  $T(e_1 + e_2 + e_3)$ , and  $T(7e_3)$ .

c) Find the nullspace of  $T$ , find a basis in  $\text{nullspace}(T)$  and find its dimension.

d) Is  $T$  one-to-one? How does this relate to the nullspace of  $T$ ?

e) Is  $T$  onto? Find  $\text{rank}(T)$ .

15. Determine which of the following are linear transformations. In case the transformation is linear, find its matrix.

a)  $T(x, y, z) = (x, x + y + z)$ .

b)  $T(x, y, z) = (1, 1)$ .

c)  $T(x, y) = (x + 1, y)$ .

16. For which values of  $t$  are the following sets linearly independent?

a)  $S = \{(t, 0, 0), (0, 1, 0), (0, 0, 1)\}$ .

b)  $S = \{(t, t, t), (t, 1, 0), (t, 0, 1)\}$ .

17. Determine whether the set  $S = \{x^2 - 2x, x^2 - 4, 2x + 5\}$  spans  $P_2$ .

18. Find the rank of the matrix  $A_n = \begin{bmatrix} 1 & 2 & 3 & \dots & n \\ n+1 & n+2 & n+3 & \dots & 2n \\ 2n+1 & 2n+2 & 2n+3 & \dots & 3n \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ n^2-n+1 & n^2-n+2 & n^2-n+3 & \dots & n^2 \end{bmatrix}$  for

$n=2,3$  and  $n=4$ . Can you find a pattern in these ranks?

19. Find (a) the rank of the matrix, (b) a basis for the nullspace, (c) a basis for the row space, (d) a basis for the column space.

$$A = \begin{bmatrix} -2 & -4 & 4 & 5 \\ 3 & 6 & -6 & -4 \\ -2 & -4 & 4 & 9 \end{bmatrix}$$