Math 3260

Final Practice – Answer key

1. Write the augmented matrix of the system, and perform two row equivalent operations, namely $R_2 - R_1 \to R_2, R_3 - kR_1 \to R_3$. We obtain

$$A \sim \begin{bmatrix} 1 & 1 & k & 3 \\ 0 & k-1 & 1-k & -1 \\ 0 & 1-k & 1-k^2 & 1-3k \end{bmatrix}.$$

Now, if $k = 1$, the matrix of the system will have rank 1, whereas the augmented matrix has rank 2 (notice that the entry in row 3 column 4 is -2 for k=1), and therefore the system will be inconsistent.
If $k \neq 1$, then we can further reduce the matrix. Add the second row to the third one and then divide the second row by $k-1$, to obtain:

$$A \sim \begin{bmatrix} 1 & 1 & k & 3 \\ 0 & 1 & -1 & \dfrac{-1}{k-1} \\ 0 & 0 & 2-k-k^2 & -3k \end{bmatrix}.$$

Now, if $2-k-k^2 = 0 \Leftrightarrow k = -2, k = 1$. But we assumed that $k \neq 1$, so if $k = -2$, the matrix of the system has rank 2, whereas the augmented matrix will have rank 3 (again the entry in row 3 column 4 is nonzero), the system is inconsistent.
If $k \neq 1$, and $k \neq -2$, the matrix of the system will have rank 3 and the system will have a unique solution.
In conclusion, the only case in which the system will not have a unique solution is when it will be inconsistent.

2. We need to write an equation in *x* and *y* that is independent of *t*. This equation turns out to be $y = 3x - 4 \Leftrightarrow 3x - y = 4$, which represents a line in the plane. I will skip the graph at this moment.

3. Find the row echelon form of the augmented matrix:

$$A \sim \begin{bmatrix} 1 & 5 & 1 & 0 \\ 0 & 1 & -2 & 0 \\ 0 & 0 & b-2+2(a-10) & c \end{bmatrix}.$$

If $b + 2a - 22 \neq 0$, then the system will have a unique solution.
If $b + 2a - 22 = 0$ and $c = 0$ then the matrix of the system has rank 2, the same as the augmented matrix, and the system will have infinitely many solutions.
If $b + 2a - 22 = 0$ and $c \neq 0$ then the matrix of the system has rank 2, whereas the augmented matrix has rank 3, and the system will be inconsistent.

4. This is fairy easy. Write $p(x) = ax^2 + bx + c$ and since the given points are on the graph of the function we have $p(2) = 5, p(3) = 2, p(4) = 5$. Then solve for *a*,*b* and *c*.

5. a) A and C are skew-symmetric, B is symmetric.
   b) Since for a skew-symmetric matrix we have $a_{ij} = -a_{ji}$, it follows that for the diagonal entries we must have $a_{ii} = -a_{ii} = 0$.
   c) Assume now that A and B are skew-symmetric matrices. We have $(AB)^T = B^T A^T = (-B)(-A) = BA$. So the question is whether $AB = BA$ (for AB to be symmetric) or $AB = -BA$ (for AB to be skew-symmetric). In general neither is true since the matrix multiplication is not commutative. (example to follow).

6. Let $A = \begin{bmatrix} u & v \\ x & y \end{bmatrix}$. If A satisfies the equation in the problem we must have:

$\begin{cases} au + cv = b \\ bu + dv = d \end{cases}$ and $\begin{cases} ax + cy = a \\ bx + dy = c \end{cases}$, for all a,b,c, and d.

From the second equation in the first system we get $v = 1, u = 0$, but these values do not satisfy the first equation (it will follow that $c = b$, but they are arbitrary) and therefore the matrix A cannot exist.

7. Let's take the transpose of the equation: $(A^2 - 5A + 6I)^T = 0^T = 0$. Applying the properties of the transpose, we get: $(A^T)^2 - 5A^T + 6I = 0$, and therefore A^T satisfies the same equation.

8. Denote $C = A + B,\ D = I + BA^{-1}$.

Assume C is invertible. Then since $D = CA^{-1}$ and A is invertible, it follows that $D^{-1} = AC^{-1}$, and therefore it is invertible.
Assume now that D is invertible. We have $C = DA$, and it is invertible since $C^{-1} = A^{-1}D^{-1}$.
In conclusion, we showed that C is invertible if and only if D is invertible, so the two matrices are both invertible or both not invertible.

9. The matrix A will not be invertible if *det*(A)=0. Using the properties of determinants, the determinant of A will remain unchanged if we perform the following row equivalent operations: $R_2 - 3R_1 \to R_2, R_3 - kR_1 \to R_3$. We have:

$|A| = \begin{vmatrix} 1 & 2 & 4 \\ 0 & -5 & -2 \\ 0 & 3-2k & 2-4k \end{vmatrix} = -5(2-4k) + 2(3-2k) = 16k - 4.$ Therefore A will fail to be invertible if $k = 1/4$.

10. Using the properties of determinants, if we add the second row to the first row the value of the determinant will be the same. We obtain:

$\begin{vmatrix} b+c & c+a & a+b \\ a & b & c \\ 1 & 1 & 1 \end{vmatrix} = \begin{vmatrix} a+b+c & c+a+b & a+b+c \\ a & b & c \\ 1 & 1 & 1 \end{vmatrix} = 0$, since the last determinant has two proportional rows (the first row is an $a+b+c$ multiple of the third one.

11. Since T is a linear transformation, we have $T(2 \cdot 0_{R^n}) = 2T(0_{R^n})$, but on the other hand $T(2 \cdot 0_{R^n}) = T(0_{R^n})$. Therefore $T(0_{R^n}) = 2T(0_{R^n})$, and by adding the opposite of $T(0_{R^n})$ on both sides of the above equation we obtain: $T(0_{R^n}) + (-T(0_{R^n})) = 0_{R^m} = 2T(0_{R^n}) + (-T(0_{R^n})) = T(0_{R^n})$. QED

Note: Someone, when asked "How do you know when a proof ends?" answered; "I see QED" which comes from the Latin (I hope I still know the spelling) "Quod erat demonstrandum", meaning, "What had to be proved".

12. a) $T(e_1), T(e_2), T(e_3)$ are the first, second and third, columns of the transformation matrix, respectively.

b) $T(e_1 + e_2 + e_3) = T(e_1) + T(e_2) + T(e_3) = (2, 5, 6)$, $T(7e_3) = 7\,T(e_3) = (0, 14, -21)$.

c) In order to find the nullspace of *T*, we find the row echelon form of *T*:

$\begin{bmatrix} 1 & -3 & 0 \\ 0 & 1 & 16 \\ 0 & 0 & 1 \end{bmatrix}$ (don't worry if you don't get the same form, as you know it is not unique).

What is important is that *rank*(*T*)=3, and therefore $nullspace(T) = \{(0,0,0)\}$, and its dimension is 0.

d) Since *rank*(*T*)=3=*dim*(R^3) (the domain of the transformation), it follows that the transformation is one-to-one.

An alternate argument is the following: Let's start with $T(u) = T(v)$. If this happens **only** when $u = v$, then T is one-to-one. But $T(u) = T(v) \Leftrightarrow T(u - v) = 0$, (or $u - v \in nullspace(T) = \{(0,0,0)\}$. Therefore $u - v = (0,0,0)$, so $u = v$.

e) *T* is onto if $T(R^3) = R^3$, or equivalently, if for any vector $b \in R^3$, there exists a vector $x \in R^3$ (called the preimage of *b*) such that $T(x) = b$.

Notice that the statement is equivalent to saying that the system $T(x) = b$ is consistent for all possible vectors *b*, or, that $b \in Column\ \ space(T)$. This last statement is equivalent to $rank(T) = 3$.

Since $rank(T) = 3$, the transformation is onto.

13. a) *T* is a linear transformation. Let $(x, y, z), (x', y', z') \in R^3, c \in R$. We have:

$$T((x, y, z) + (x', y', z')) = T(x + x', y + y', z + z') = (x + x', x + x' + y + y' + z + z')$$
$$= (x, x + y + z) + (x', x' + y' + z') = T(x, y, z) + T(x', y', z')$$

$$T(c(x, y, z)) = T(cx, cy, cz) = (cx, cx + cy + cz) = c(x, x + y + z) = cT(x, y, z)$$

b) *T* is not a linear transformation since $(1,1) = T(2,2,2) \neq 2T(1,1,1) = (2,2)$.

c) T is not a linear transformation since if it were we would have $(1,0) = T(0,0) = T((0,0)+(0,0)) = T(0,0) + T(0,0) = (2,0)$.

14. a) It can be easily seen that for any $t \neq 0$, the rank of the matrix formed with the three vectors is 1, therefore S is linearly independent for all $t \neq 0$.

b) Let us consider the matrix $A = \begin{bmatrix} t & t & t \\ t & 1 & 0 \\ t & 0 & 1 \end{bmatrix} \sim \left\{ \begin{bmatrix} t & t & t \\ 0 & 1-t & -t \\ 0 & -t & 1-t \end{bmatrix} \right\}$. Now, rather than trying to reduce it further I will consider the determinant of the matrix. The set S is linearly independent if and only if the matrix will have rank 3, which is equivalent to its determinant being nonzero. Since $\det(A) = t[(1-t)^2 - t^2] = t(1-2t)$, it follows that $\det(A) = 0 \Leftrightarrow t = 0, t = 1/2$.

Therefore, S is linearly independent if and only if $t \neq 0, t \neq 1/2$.

15. Let $a(x^2 - 2x) + b(x^2 - 4) + c(2x+5) = 0 \Leftrightarrow x^2(a+b) + x(-2a+2c) - 4b + 5c = 0$ for **all** x, therefore: $\begin{cases} a + b = 0 \\ -2a + 2c = 0 \\ -4b + 5c = 0 \end{cases} \Rightarrow a = b = c = 0$, and therefore the set is linearly independent.

16. For $n = 2,\ A_2 = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix},\ rank(A_2) = 2$.

For $n = 3,\ A_3 = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix} \overset{\substack{R_3 - R_2 \to R_3 \\ R_2 - R_1 \to R_2}}{\sim} \begin{bmatrix} 1 & 2 & 3 \\ 3 & 3 & 3 \\ 3 & 3 & 3 \end{bmatrix},\ rank(A_3) = 2$.

In general, in the matrix of order n, if we do the row equivalent operations: $R_n - R_{n-1} \to R_n, R_{n-1} - R_{n-2} \to R_{n-1}, \ldots, R_2 - R_1 \to R_2$ we get:

$A_n = \begin{bmatrix} 1 & 2 & 3 & \ldots & n \\ n & n & n & \ldots & n \\ n & n & n & \ldots & n \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ n & n & n & \cdots & n \end{bmatrix}$, and therefore $rank(A_n) = 2$.

17. The row echelon form of A is $\begin{bmatrix} 1 & 2 & -2 & 5/2 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$. Therefore $rank(A) = 2$.

We have $nullspace(A) = \{(2t - 2s, s, t, 0) : s, t \in R\}$, so a basis in the nullspace(A) is $\{(2,0,1,0), (-2,1,0,0)\}$.

$Row\ space(A) = span\{(1,2,-2,5/2), (0,0,0,1)\}$, $Column\ space(A) = span\{(-2,3,-2), (5,-4,9)\}$