Lines and Planes in 3D

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Recall that given a point \( P = (a, b, c) \), one can draw a vector from the origin to \( P \). Such a vector is called the **position vector** of the point \( P \) and its coordinates are \( \langle a, b, c \rangle \), the same as \( P \). Position vectors are usually denoted \( \vec{r} \).

In this section, we derive the equations of lines and planes in 3-D. We do so by finding the conditions a point \( P = (x, y, z) \) or its corresponding position vector \( \vec{r} = \langle x, y, z \rangle \) must satisfy in order to belong to the object being studied (line or plane).
In 3-D, like in 2-D, a line is uniquely determined when one point on the line and the direction of the line are given. In this section, we assume we are given a point \( P_0 = (x_0, y_0, z_0) \) on the line and a direction vector \( \vec{v} = \langle a, b, c \rangle \). Our goal is to determine the equation of the line \( L \) which goes through \( P_0 \) and is parallel to \( \vec{v} \).

**Definition**

\( a, b, \) and \( c \) are called the direction numbers of the line \( L \).

Let \( P(x, y, z) \) be an arbitrary point on \( L \). We wish to find the conditions \( P \) must satisfy to be on the line \( L \).
Lines in 3D

Figure: Line through \( P_0 \) parallel to \( \vec{v} \)

\[ P_0(x_0, y_0, z_0) \]

\[ P(x, y, z) \]

\[ \vec{r}_0 \]

\[ \vec{r} \]

\[ \vec{v} \]
Looking at the figure on the previous slide, we see that a necessary and sufficient condition for the point $P$ to be on the line $L$ is that $\overrightarrow{P_0P}$ be parallel to $\overrightarrow{v}$.

This gives us

$$\overrightarrow{r} = \overrightarrow{r_0} + t \overrightarrow{v}$$

**Definition**

The above equation is known as the *vector equation* of the line $L$. The scalar $t$ used in the equation is called a *parameter*.

The parameter $t$ can be any real number. As it varies, the point $P$ moves along the line. When $t = 0$, $P$ is the same as $P_0$. When $t > 0$, $P$ is away from $P_0$ in the direction of $\overrightarrow{v}$ and when $t < 0$, $P$ is away from $P_0$ in the direction opposite $\overrightarrow{v}$. The larger $t$ is (in absolute value), the further away $P$ is from $P_0$. 
If we switch to coordinates, the vector equation becomes
\[ \langle x, y, z \rangle = \langle x_0, y_0, z_0 \rangle + t \langle a, b, c \rangle. \] Two vectors are equal when their corresponding coordinates are equal. Thus, we obtain

\[
\begin{align*}
    x &= x_0 + at \\
    y &= y_0 + bt \\
    z &= z_0 + ct
\end{align*}
\]

**Definition**

The above equation is known as the **parametric equation** of the line \( L \).
If we solve for $t$ in the parametric equation, assuming that $a \neq 0$, $b \neq 0$, and $c \neq 0$ we obtain

\[
\frac{x - x_0}{a} = \frac{y - y_0}{b} = \frac{z - z_0}{c}
\]

**Definition**

The above equations are known as the **symmetric equations** of the line $L$.

If $a = 0$, we only solve for $t$ in the last two parametric equations to get

\[
\frac{x}{x_0} = \frac{y - y_0}{b} = \frac{z - z_0}{c}
\]

We would obtain a similar result if one of the other direction numbers is 0.
Example

Find the parametric and symmetric equations of the line through $P(-1, 4, 2)$ in the direction of $\mathbf{v} = \langle 1, 2, 3 \rangle$.

Example

Find the parametric and symmetric equations of the line through $P_1(1, 2, 3)$ and $P_2(2, 4, 1)$. 
We can also derive a formula for the equation of a line given two points on the line: \( P_0 = (x_0, y_0, z_0) \) and \( P_1 = (x_1, y_1, z_1) \). Using \( P_0 \) for the point and \( \overrightarrow{P_0P_1} \) for the direction vector, we get:

\[
\begin{align*}
    x &= x_0 + t(x_1 - x_0) \\
    y &= y_0 + t(y_1 - y_0) \\
    z &= z_0 + t(z_1 - z_0)
\end{align*}
\]

Factoring differently gives

\[
\begin{align*}
    x &= (1 - t)x_0 + tx_1 \\
    y &= (1 - t)y_0 + ty_1 \\
    z &= (1 - t)z_0 + tz_1
\end{align*}
\]

When \( t = 0 \), we are at the point \( P_0 \) and when \( t = 1 \), we are at the point \( P_1 \). So, if \( t \) is allowed to take on any real value, then this equation will describe the whole line. On the other hand, if we restrict \( t \) to \([0, 1]\), then this equation describes the portion of the line between \( P_0 \) and \( P_1 \) which is called the **line segment** from \( P_0 \) to \( P_1 \).
Example

Find the equation of the line segment from \( P_0 = (1, 2, 3) \) to \( P_1 (2, 4, 1) \).
Recall that in 2-D two lines were either parallel or intersected. In 3-D it is also possible for two lines to not be parallel and to not intersect. Such lines are called **skew lines**.
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Two lines are parallel if their direction vectors are parallel.

If two lines are not parallel, we can find if they intersect if there exists values of the parameters in their equations which produce the same point. More specifically, if the first line has equation \( \vec{r_0} + t \vec{v} \) and the second line has equation \( \vec{R_0} + s \vec{u} \) then they will intersect if there exists a value for \( t \) and \( s \) such that \( \vec{r_0} + t \vec{v} = \vec{R_0} + s \vec{u} \).
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Also, when two lines intersect, we can find the angle between them by finding the smallest angle between their direction vectors (using the dot product).
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Also, when two lines intersect, we can find the angle between them by finding the smallest angle between their direction vectors (using the dot product).

Finally, two lines are perpendicular if their direction vectors are perpendicular.
**Example**

Let $L_1$ be the line through $(1, -6, 2)$ with direction vector $\langle 1, 2, 1 \rangle$ and $L_2$ be the line through $(0, 4, 1)$ with direction vector $\langle 2, 1, 2 \rangle$. Determine if the lines are parallel, if they intersect or if they are skew. If they intersect, find the point at which they intersect.

**Example**

Find the angle between the two previous lines.

**Example**

Find the points at which $L_1$ in the example above intersects with the coordinate planes.
Be able to find the equation of a line given a point and a direction or given two points.

Be able to tell if two lines are parallel, intersect or are skewed.

Be able to find the angle between two lines which intersect.

Be able to find the points at which a line intersect with the coordinate planes.
A plane is uniquely determined given a point on the plane and a vector perpendicular to the plane. Such a vector is said to be **normal** to the plane. To help visualize this, consider the figure below.

**Figure:** Plane determined by a point and its normal
Planes in 3D: Equation Given a Point and a Vector Perpendicular

Given a point \( P_0 = (x_0, y_0, z_0) \) and a normal \( \vec{n} = \langle a, b, c \rangle \) to a plane, a point \( P = (x, y, z) \) will be on the plane if \( P_0 \vec{P} \) is perpendicular to \( \vec{n} \) that is \( \vec{n} \cdot P_0 \vec{P} = 0 \).

This is known as the **vector equation** of a plane. Switching to coordinates, we get

\[
ax + by + cz + d = 0
\]

where \( d = -ax_0 - by_0 - cz_0 \). This is known as the **scalar equation** of a plane. It is also the equation of a plane in implicit form.

**Fact**

*Note that when we know the scalar equation of a plane, we automatically know its normal; it is given by the coefficients of \( x, y, \) and \( z \).*

**Fact**

*A plane in 3D is the analogous of a line in 2D.*
Example
What is the normal to the plane $3x + 2y - z = 10$?

Example
What is the scalar equation of a plane through $(1, 2, 3)$ with normal $\langle 2, 1, 4 \rangle$?

Example
Find the equation of the plane through the points $P_1(0, 1, 1)$, $P_2(1, 0, 1)$ and $P_3(1, -3, -1)$.
Two planes are parallel if and only if their normals are parallel.
Two planes are parallel if and only if their normals are parallel.

If two planes $p_1$ with normal $\vec{n}_1$ and $p_2$ with normal $\vec{n}_2$ are not parallel, then the angle $\theta$ between them is defined to be the smallest angle between their normals that is the angle with the non-negative cosine. In other words,

$$\theta = \cos^{-1} \left( \frac{\vec{n}_1 \cdot \vec{n}_2}{\|\vec{n}_1\| \|\vec{n}_2\|} \right)$$

(1)
Example

Find the angle between the planes $p_1 : x + y + z - 1 = 0$ and $p_2 : x - 2y + 3z - 1 = 0$. Find their intersection.
Planes in 3D: Summary for Planes

In addition, using the material studied so far, you should be able to do the following:

1. Be able to find the equation of a plane given a point on the plane and a normal to the plane.
2. Be able to find the equation of a plane given three points on the plane.
3. Be able to find the equation of a plane through a point and parallel to a given plane.
4. Be able to find the equation of a plane through a point and a line not containing the point.
5. Be able to tell if two planes are parallel, perpendicular.
6. Be able to find the angle between two planes.
7. Be able to find the traces of a plane.
8. Be able to find the intersection of two planes.
Exercises

See the problems at the end of my notes on equations of lines and planes in 3D.