Proof by Induction

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The purpose of this section is to study the proof technique known as \textbf{mathematical induction}. Before we do so, we will quickly review the other proof techniques used in mathematics.
1 Direct Proof: We derive the result to prove by combining logically the given assumptions (if any), definitions, axioms and known theorems.
Proof Techniques Other than Induction

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2. **Proof by Contrapositive:** Suppose that $P$ and $Q$ are two statements. "If $P$ then $Q$" is equivalent to its contrapositive "if not $Q$ then not $P$ ". Instead of proving one, the other can be proven.
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4. **Proof by Exhaustion**: We divide the result to prove into cases and prove each one separately.
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5. **Proof by Construction**: We prove an object having certain properties exists by constructing an example of an object with the required properties.
Proofs by induction are often used when one tries to prove a statement made about natural numbers or integers. Here are examples of statements where induction would be used.

- For every natural number \( n \), \( 1 + 2 + 3 + \ldots + n = \frac{n(n + 1)}{2} \)
- If \( x > -1 \), and \( n \) is a natural number, then \( (1 + x)^n \geq 1 + nx \)
Let $P(n)$ denote a statement about natural numbers with the following properties:

1. The statement is true when $n = 1$ i.e. $P(1)$ is true.
2. $P(k + 1)$ is true whenever $P(k)$ is true for any positive integer $k$.

Then $P(n)$ is true for all $n \in \mathbb{N}$.
1st Principle of induction

Remarks

- The case \( n = 1 \) is called the **base case**.
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The principle of mathematical induction is also true if instead of starting at 1, we start at any integer \( n_0 \). In other words, if we prove that \( P(n_0) \) is true and \( P(k + 1) \) is true whenever \( P(k) \) is true, \( k \geq n_0 \), then \( P(n) \) will be true for all \( n \in \mathbb{Z} \) such that \( n \geq n_0 \).
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The principle of mathematical induction is also true if instead of starting at 1, we start at any integer $n_0$. In other words, if we prove that $P(n_0)$ is true and $P(k + 1)$ is true whenever $P(k)$ is true, $k \geq n_0$, then $P(n)$ will be true for all $n \in \mathbb{Z}$ such that $n \geq n_0$.

When doing a proof by induction, it is important to write explicitly what the statement $P(n)$ is so we know what we have to prove for a given $n$. Before proving $P(1)$, write clearly what $P(1)$ says. Similarly, when we assume $P(k)$ true and want to deduce $P(k + 1)$, write clearly what both $P(k)$ and $P(k + 1)$ say so we know what we are assuming and what we need to prove.
1st Principle of induction

Examples

**Example**

If $n$ is a natural number, then $1 + 2 + 3 + \ldots + n = \frac{n(n + 1)}{2}$

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(known as Bernoulli’s inequality). If $x > -1$, and $n$ is a natural number, then $(1 + x)^n \geq 1 + nx$
Proving that $P(1)$ is true is essential. Consider the statement $n + 1 = n$ for all $n \geq 0$. This is obviously false. However, if we do not bother to check whether $P(1)$ is true and we assume that $P(k)$ is true then we can prove that $P(k + 1)$ is also true. $P(k + 1)$ says that $n + 2 = n + 1$.

\[
\begin{align*}
  n + 2 & = n + 1 + 1 \\
         & = n + 1 \text{ by assumption since } n + 1 = n
\end{align*}
\]
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More Remarks

- Proving that $P(1)$ is true is essential. Consider the statement $n + 1 = n$ for all $n \geq 0$. This is obviously false. However, if we do not bother to check whether $P(1)$ is true and we assume that $P(k)$ is true then we can prove that $P(k+1)$ is also true. $P(k+1)$ says that $n + 2 = n + 1$.

\[ n + 2 = n + 1 + 1 = n + 1 \text{ by assumption since } n + 1 = n \]

- Sometimes, it is not easy to deduce that $P(k+1)$ is true knowing that $P(k)$ is true, especially if we do not have a relationship between $P(k)$ and $P(k+1)$. In such cases, another form of mathematical induction can be used.
2nd Principle of induction
Definition

**Theorem**

Let $P(n)$ denote a statement about natural numbers with the following properties:

1. The statement is true when $n = 1$ i.e. $P(1)$ is true.
2. $P(k)$ is true whenever $P(j)$ is true for all positive integers $1 \leq j < k$. Then $P(n)$ is true for every natural number.
Consider $f : \mathbb{N} \to \mathbb{R}$ defined by $f(1) = 0$, $f(2) = \frac{1}{3}$, and for $n > 2$ by $f(n) = \frac{n - 1}{n + 1}f(n - 2)$. By computing values of $f(n)$ for $n = 3, 4, 5, 6$, give a conjecture as to what a direct formula for $f$ might be. Prove your conjecture by induction.
See the problems at the end of my notes on proof by induction.