2.3 BOUNDED FUNCTIONS

2.3 Bounded Functions

2.3.1 Suprema and Infima of a Function

In the previous sections, we have studied the notions of maximum, minimum, being bounded, supremum and infimum in the context of sets. These notions can also be applied to functions. In this case, they are applied to the range of a function. More specifically, we have the following definitions.

Definition 2.3.1 (Bounded Functions) Let \( f : D \rightarrow \mathbb{R} \).

1. \( f \) is said to be bounded above if its range is bounded above.
2. \( f \) is said to be bounded below if its range is bounded below.
3. \( f \) is said to be bounded if its range is both bounded above and below.
4. If the range of \( f \) has a maximum (largest) element, then that element is called the maximum of \( f \) and is denoted \( \max f \).
5. If the range of \( f \) has a minimum (smallest) element, then that element is called the minimum of \( f \) and is denoted \( \min f \).

Definition 2.3.2 (Supremum and Infimum) Let \( f : D \rightarrow \mathbb{R} \) and assume that \( D \neq \emptyset \).

1. The supremum of \( f \), denoted \( \sup f \), is defined by
   \[
   \sup f = \sup \{ f(x) : x \in D \}
   \]
   In other words, it is the supremum of the range of \( f \).
2. The infimum of \( f \), denoted \( \inf f \), is defined by
   \[
   \inf f = \inf \{ f(x) : x \in D \}
   \]
   In other words, it is the infimum of the range of \( f \).

Remark 2.3.3 If \( f : D \rightarrow \mathbb{R} \) is bounded on a non-empty set \( D \) then the following inequality should be clear for every \( x \in D \)

\[
\inf f \leq f(x) \leq \sup f
\]

Example 2.3.4 Consider \( f : (0, 2) \rightarrow \mathbb{R} \) defined by \( f(x) = x^2 \). The range of \( f \) is \((0, 4)\) (why?). Therefore, \( f \) is bounded on \((0, 2)\). However, \( f \) does not have a maximum or a minimum on \((0, 2)\). However, \( \sup f = 4, \inf f = 0 \). If we replace \((0, 2)\) by \([0, 2]\), then \( f \) is bounded on \([0, 2]\) and it also has a maximum (hence a supremum) equal to 4 and a minimum (hence an infimum) equal to 0.
Example 2.3.5 Consider $g : (0, 1) \rightarrow \mathbb{R}$ defined by $g(x) = \frac{1}{x}$. The range of $g$ is $(1, \infty)$. Therefore, $g$ is not bounded above on $(0, 1)$ but it is bounded below. $g$ does not have a maximum or a minimum on $(0, 1)$. However, $\inf f = 1$. If we replace $(0, 1)$ by $(0, 1]$, then the range of $g$ is $[1, \infty)$. So, we see that $g$ is bounded below and has a minimum which is also the infimum equal to 1. But $g$ is not bounded above.

Example 2.3.6 Consider $h : \mathbb{R} \rightarrow \mathbb{R}$ defined by $h(x) = \sin(x)$. The range of $h$ is $[-1, 1]$. So, $h$ is bounded, has a minimum, $-1$ and a maximum, 1. Also $\sup f = 1$ and $\inf f = -1$.

Remark 2.3.7 You may have noticed in this section, or in books about mathematics that we often say that a function has a certain property on a set. For example we say that $\ln x$ is continuous on $(0, \infty)$. We may also say that $x^2$ is increasing on $[0, \infty)$. What we mean when we say $f$ has a certain property on a set $D$ is that $f$ has that property for every $x$ in $D$.

Example 2.3.8 Prove that if $f$ and $g$ are bounded above on a non-empty set $S$ then $\sup (f + g) \leq \sup f + \sup g$.

The fact that both $f$ and $g$ are bounded above on a nonempty set $S$ implies that $\sup f$ and $\sup g$ exist by the completeness axiom. If $x$ is an arbitrary element in $S$ then $f(x) \leq \sup f$ and $g(x) \leq \sup g$. By definition, $(f + g)(x) = f(x) + g(x)$ therefore, $(f + g)(x) \leq \sup f + \sup g$. This means that $\sup f + \sup g$ is an upper bound for $f + g$. Hence $f + g$ is bounded above. It follows that $\sup (f + g)$ exists by the completeness axiom. Since $\sup f + \sup g$ is an upper bound of $f + g$ and $\sup (f + g)$ is the smallest upper bound of $f + g$, we must have $\sup (f + g) \leq \sup f + \sup g$.

2.3.2 Exercises

1. Consider $f : (0, \infty) \rightarrow \mathbb{R}$ defined by $f(x) = \frac{1}{x}$.
   
   (a) Prove that $f$ is not bounded above.
   
   (b) Prove that the restriction of $f$ to $[\delta, \infty)$ is bounded for every number $\delta > 0$.

2. Let $f$ be an increasing function on a closed interval $[a, b]$. Make a conjecture regarding whether $f$ has a maximum and a minimum on $[a, b]$ then prove your conjecture.

3. Prove that if $f$ and $g$ are bounded above on a non-empty set $S$ then $\sup (f + g) \leq \sup f + \sup g$.

4. Prove that if $f$ and $g$ are bounded below on a non-empty set $S$ then $\inf (f + g) \geq \inf f + \inf g$. 
5. Give an example of two bounded function on $[0, 1]$ such that $\sup (f + g) < \sup f + \sup g$.

6. Prove that if $f$ and $g$ are bounded above on a non-empty set $S$ and $f(x) \leq g(x)$ on $S$ then $\sup f \leq \sup g$.

7. Prove that if $f$ is a bounded function on a non-empty set $S$ then

$$\sup (cf) = \begin{cases} c \sup f & \text{if } c > 0 \\ c \inf f & \text{if } c < 0 \end{cases}$$

8. Prove that if $f$ is a bounded function on a non-empty set $S$ then

$$|\sup f| \leq \sup |f|$$
2.3.3 Hints for the Exercises

For all the problems, remember to first clearly state what you have to prove or do.

1. Consider \( f : (0, \infty) \to \mathbb{R} \), \( x \mapsto \frac{1}{x} \).
   
   (a) Prove that \( f \) is not bounded above.
   
   **Hint:** for an arbitrary \( r \in \mathbb{R} \), look at \( f \left( \frac{1}{1+r} \right) \).

   (b) Prove that the restriction of \( f \) to \([\delta, \infty)\) is bounded for every number \( \delta > 0 \).
   
   **Hint:** what is \( f ([\delta, \infty)) \)?

2. Let \( f \) be an increasing function on a closed interval \([a, b]\). Make a conjecture regarding whether \( f \) has a maximum and a minimum on \([a, b]\) then prove your conjecture.
   
   **Hint:** what is \( f ([a, b]) \)?

3. Prove that if \( f \) and \( g \) are bounded above on a non-empty set \( S \) then
   
   \[ \sup (f + g) \leq \sup f + \sup g \]
   
   **Hint:** recall that the function \( f + g \) is defined by \( (f + g)(x) = f(x) + g(x) \). Use the definition of supremum and the fact that for any function \( f \), \( f(x) \leq \sup f \) for every \( x \) in the domain of \( f \).

4. Prove that if \( f \) and \( g \) are bounded below on a non-empty set \( S \) then
   
   \[ \inf (f + g) \geq \inf f + \inf g \]
   
   **Hint:** similar to the previous problem.

5. Give an example of two bounded function on \([0, 1]\) such that \( \sup (f + g) < \sup f + \sup g \).
   
   **Hint:** think of functions where the supremum does not happen at the same value of \( x \).

6. Prove that if \( f(x) \leq g(x) \) on \( S \) then \( \sup f \leq \sup g \).
   
   **Hint:** use the definition of supremum and the fact that for any function \( f \), \( f(x) \leq \sup f \) for every \( x \) in the domain of \( f \).

7. Prove that if \( f \) is a bounded function on a non-empty set \( S \) then
   
   \[ \sup (cf) = \begin{cases} c \sup f & \text{if } c > 0 \\ c \inf f & \text{if } c < 0 \end{cases} \]
   
   **Hint:** use the definition of supremum and infimum and the fact that for any function \( f \), \( \inf f \leq f(x) \leq \sup f \) for every \( x \) in the domain of \( f \).
8. Prove that if $f$ is a bounded function on a non-empty set $S$ then

$$|\sup f| \leq \sup |f|$$

Hint: use the definition of supremum. Also remember that for any function $f$, $-|f(x)| \leq f(x) \leq |f(x)|$ for every $x$ in the domain of $f$ and finally to prove an inequality of the form $|A| \leq C$ you need to prove that $-C \leq A \leq C$. 