Abstract

3D Transformations. This handout is just a summary of the material to know regarding 3D transformations. The details as well as examples can be found in the accompanying Maple worksheet called tran3D.mws.

1 3D Transformations

We cover the basic transformations: translation, scaling and rotation. As in 2D, we will use homogeneous coordinates. That is, we will add a fourth coordinate to each point. Thus, the point of coordinates \((x, y, z)\) will become \((x, y, z, 1)\).

In 3D, there are two possible coordinate systems: the right-handed coordinate system and the left-handed coordinate system. To picture these coordinate systems, imagine holding your hand perpendicular to your arm, your thumb pointing upward. In a right-handed coordinate system, the \(y\)-axis is your right arm, the \(x\)-axis is your fingers and the \(z\)-axis is your thumb. A left-handed coordinate system is similar, but from your left arm instead. By convention, right-handed systems are the standard in mathematics.

1.1 Translation

Because we are using homogeneous coordinates, we can represent a translation as a matrix product. More precisely, if the original point is \(V(x, y, z, 1)\), and the transformed point is \(V'(x', y', z', 1)\), and if the translation is \(T_x\) units in the \(x\) direction, \(T_y\) units in the \(y\) direction and \(T_z\) units in the \(z\) direction, then we should have

\[
\begin{align*}
    x' &= x + T_x \\
    y' &= y + T_y \\
    z' &= z + T_z
\end{align*}
\]  

This can be represented by the matrix product

\[
V' = TV
\]
where
\[ V' = \begin{bmatrix} x' \\ y' \\ z' \\ 1 \end{bmatrix} \]
\[ V = \begin{bmatrix} x \\ y \\ z \\ 1 \end{bmatrix} \]
and
\[ T = \begin{bmatrix} 1 & 0 & 0 & T_x \\ 0 & 1 & 0 & T_y \\ 0 & 0 & 1 & T_z \\ 0 & 0 & 0 & 1 \end{bmatrix} \]

The reader will verify that equation 2 gives us equation 1.
The reader will also verify that \( T^{-1} \) is obtained from \( T \) by negating \( T_x \), \( T_y \) and \( T_z \). (See problems at the end of the document).

### 1.2 Scaling

As in 2D, scaling is a transformation of the form
\[
\begin{align*}
x' &= xS_x \\
y' &= yS_y \\
z' &= zS_z
\end{align*}
\]
(3)

It can be represented by the matrix product
\[ V' = SV \]
(4)

where
\[ S = \begin{bmatrix} S_x & 0 & 0 & 0 \\ 0 & S_y & 0 & 0 \\ 0 & 0 & S_z & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \]

The reader will verify that equation 4 gives us equation 3.
The reader will also verify that \( S^{-1} \) is obtained from \( S \) by taking the reciprocal of \( S_x \), \( S_y \) and \( S_z \).

### 1.3 Rotation

While the other two transformations are very similar to their 2D counterpart, rotation in 3D is somewhat more complicated. There are more possible axes of rotation. In this section, we will only consider rotations about one of the coordinate axes.
1. If we rotate about the $x$-axis, counterclockwise (looking along the $x$-axis towards the origin), this will only change the $y$ and $z$ coordinates. The rotation matrix, $R_x$, will be like the 2D rotation matrix with no operation on $x$.

$$R_x = \begin{bmatrix}
1 & 0 & 0 & 0 \\
0 & \cos \theta & -\sin \theta & 0 \\
0 & \sin \theta & \cos \theta & 0 \\
0 & 0 & 0 & 1
\end{bmatrix}$$

2. Similarly, if we rotate about the $y$-axis, counterclockwise (looking along the $y$-axis towards the origin), this will only change the $x$ and the $z$ coordinates. The rotation matrix, $R_y$, will be:

$$R_y = \begin{bmatrix}
\cos \theta & 0 & \sin \theta & 0 \\
0 & 1 & 0 & 0 \\
-\sin \theta & 0 & \cos \theta & 0 \\
0 & 0 & 0 & 1
\end{bmatrix}$$

3. Finally, if we rotate about the $z$-axis, counterclockwise (looking along the $z$-axis towards the origin), this will only change the $x$ and $y$ coordinates. The rotation matrix, $R_z$, will be:

$$R_z = \begin{bmatrix}
\cos \theta & -\sin \theta & 0 & 0 \\
\sin \theta & \cos \theta & 0 & 0 \\
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1
\end{bmatrix}$$

You will notice that in $R_y$, the minus sign is in a different place, this is not an error. The reader can work out the details to see that this is indeed the case.

The inverse of these rotation matrices is obtained by negating the angle of rotation. This produces the transpose of the matrix. In other words, we have

$$R_x^{-1} = R_x^T$$
$$R_y^{-1} = R_y^T$$
$$R_z^{-1} = R_z^T$$

You may ask yourself why we need to know this since the two matrices are the same. It is much easier and faster to find the transpose of a matrix than its inverse. Though $R_x^{-1} = R_x^T$, a computer will find $R_x^T$ much faster than it will find $R_x^{-1}$. In addition, because of the algorithms used, the computation of $R_x^{-1}$ is more likely to introduce a small error.

### 1.4 Final Remarks

- Like in 2D, scaling is done with respect to the origin. If an object is away from the origin, scaling it will also cause it to move. If the scaling must be
done so that a point must not move, then the object has to be translated first so that the point which must not move is at the origin. After the scaling, the object has to be translated back.

- If a rotation must be made around an axis parallel to one of the coordinate axes, the object must be translated first so that the axis of rotation becomes the coordinate axis to which it is parallel. After the object has been rotated, it must be translated back.

- All the described transformations are pointwise transformations. Because they are also affine transformations (they preserve parallel lines), we can transform objects which are made of line segments by transforming the points, then redrawing the line segments between the new points.

2 Practice Problems

1. Go over the worksheet tran3D.mws.

2. Explain why $T^{-1}$ is obtained from $T$ by negating $T_x$, $T_y$ and $T_z$. Then, prove it mathematically.

3. Prove equation 5.

4. Consider the cube whose vertices are $(1, 1, 1), (2, 1, 1), (2, 2, 1), (1, 2, 1)$ for the bottom part and $(1, 1, 2), (2, 1, 2), (2, 2, 2), (1, 2, 2)$ for the top part. Develop a Maple worksheet which does the following:

   (a) Plot the original cube.

   (b) Performs a translation by $(2, 1, 3)$ on the cube. Plot the original cube as well as its transformed version.

   (c) Rescale the original cube by $(2, 2, 3)$.

   (d) Rescale the original cube by $(2, 2, 3)$ so that the point $(1, 1, 1)$ does not move.

   (e) Rotate the original cube around an axis parallel to the $z$-axis, through $(1, 1, 1)$ by 30 degrees. Plot the original cube as well as its transformed version.

   (f) You will call your worksheet lect4pb4.mws.

3 Resources

This is a list of books and other resources I used to compile these notes.
References


