

Transformations of Functions

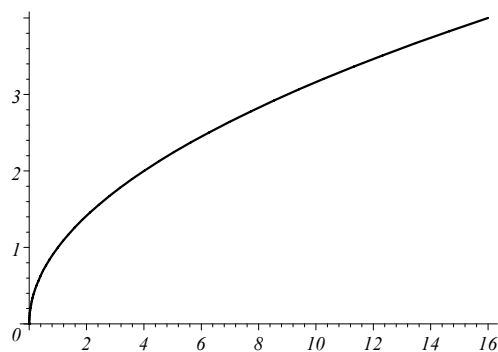
S. F. Ellermeyer

1 Horizontal and Vertical Translations

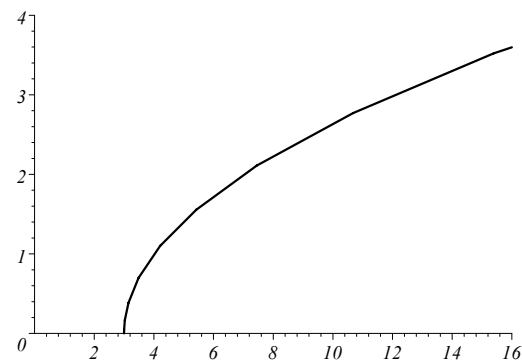
1.1 Horizontal Translations

If we begin with the formula $y = f(x)$ and we replace x with $x - a$ where a is a constant, then we get the formula $y = f(x - a)$. The graph of $y = f(x - a)$ is obtained by translating the graph of $y = f(x)$ to the **right** by a units if a is a **positive** number or to the **left** by $|a|$ units if a is a **negative** number. If the point (x, y) is on the graph of $y = f(x)$, then the point $(x + a, y)$ is on the graph of $y = f(x - a)$.

Example 1 Starting with the formula $y = \sqrt{x}$, if we replace x with $x - 3$, then we get the formula $y = \sqrt{x - 3}$. The graph of $y = \sqrt{x - 3}$ is obtained by translating the graph of $y = \sqrt{x}$ to the right by 3 units.

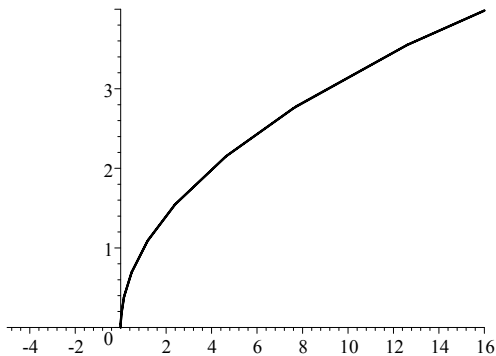


Graph of $y = \sqrt{x}$

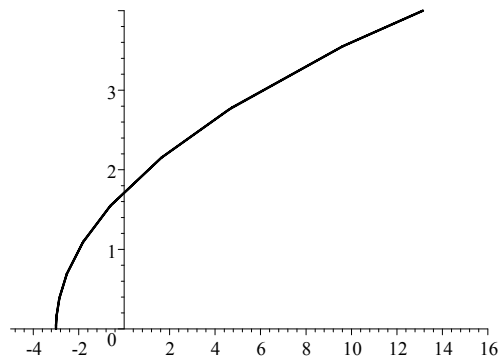


Graph of $y = \sqrt{x - 3}$

Example 2 Starting with the formula $y = \sqrt{x}$, if we replace x with $x - (-3) = x + 3$, then we get the formula $y = \sqrt{x + 3}$. The graph of $y = \sqrt{x + 3}$ is obtained by translating the graph of $y = \sqrt{x}$ to the left by 3 units.



Graph of $y = \sqrt{x}$

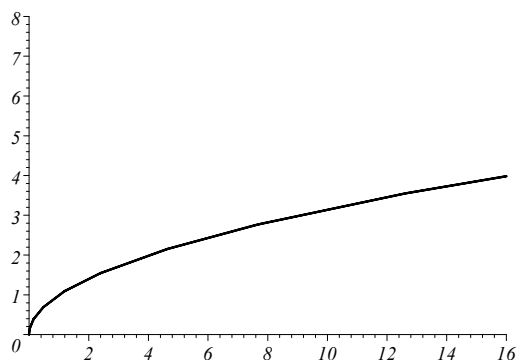


Graph of $y = \sqrt{x + 3}$

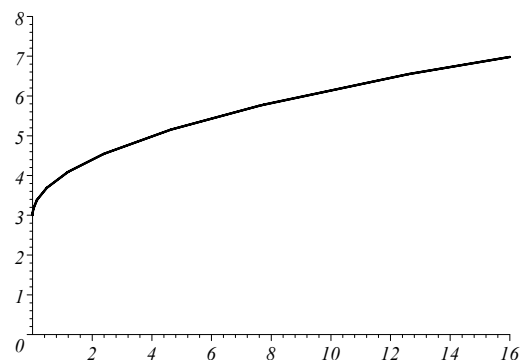
1.2 Vertical Translations

If we begin with the formula $y = f(x)$ and we replace y with $y - a$ where a is a constant, then we get the formula $y - a = f(x)$ which can also be written as $y = f(x) + a$. The graph of $y = f(x) + a$ is obtained by translating the graph of $y = f(x)$ **up** by a units if a is a **positive** number or **down** by $|a|$ units if a is a **negative** number. If the point (x, y) is on the graph of $y = f(x)$, then the point $(x, y + a)$ is on the graph of $y = f(x) + a$.

Example 3 Starting with the formula $y = \sqrt{x}$, if we replace y with $y - 3$, then we get the formula $y - 3 = \sqrt{x}$ which can also be written as $y = \sqrt{x} + 3$. The graph of $y = \sqrt{x} + 3$ is obtained by translating the graph of $y = \sqrt{x}$ up by 3 units.

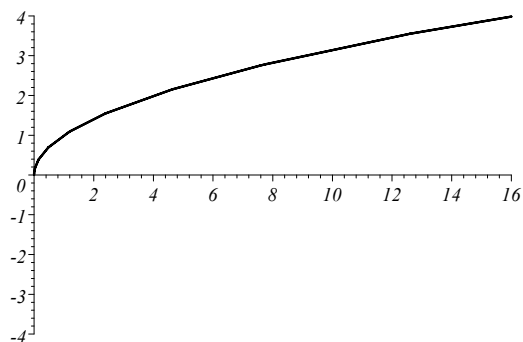


Graph of $y = \sqrt{x}$

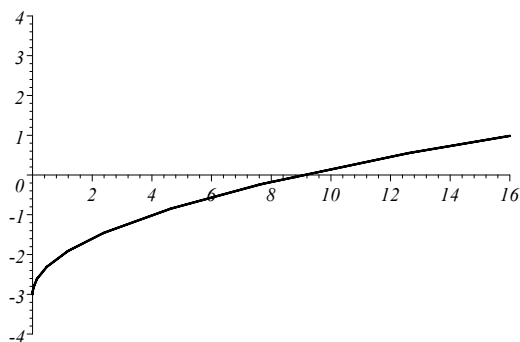


Graph of $y = \sqrt{x} + 3$

Example 4 Starting with the formula $y = \sqrt{x}$, if we replace y with $y - (-3) = y + 3$, then we get the formula $y + 3 = \sqrt{x}$ which can also be written as $y = \sqrt{x} - 3$. The graph of $y = \sqrt{x} - 3$ is obtained by translating the graph of $y = \sqrt{x}$ down by 3 units.



Graph of $y = \sqrt{x}$



Graph of $y = \sqrt{x} - 3$

Exercise 5 Perform the following transformations to the function $y = x^2$.

1. a translation to the right by 2 units.
2. a translation to the left by 2 units.

3. a translation upward by 2 units.
4. a translation downward by 2 units.

In each case, write the formula that gives the requested transformation and draw the graph of the transformed function

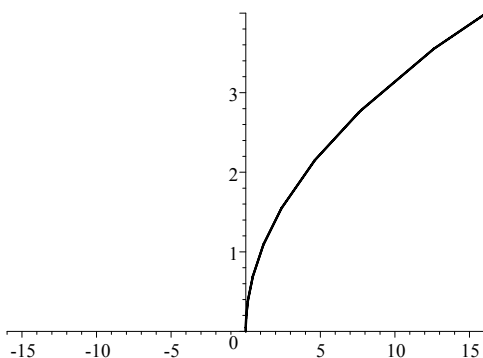
Exercise 6 Repeat the above exercise for the functions $y = -2x + 1$, $y = |x|$, and $y = 1/x$.

2 Reflections

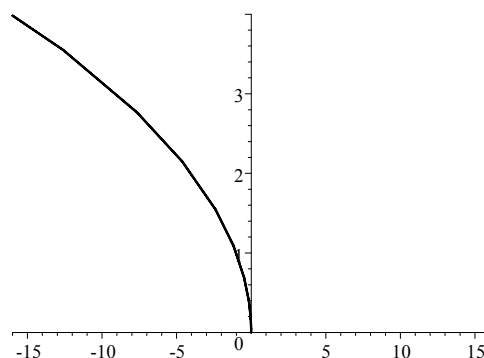
2.1 Reflections Through the y Axis

If we begin with the formula $y = f(x)$ and we replace x with $-x$, then we get the formula $y = f(-x)$. The graph of $y = f(-x)$ is obtained by reflecting the graph of $y = f(x)$ through the y axis. If the point (x, y) is on the graph of $y = f(x)$, then the point $(-x, y)$ is on the graph of $y = f(-x)$.

Example 7 Starting with the formula $y = \sqrt{x}$, if we replace x with $-x$, then we get the formula $y = \sqrt{-x}$. The graph of $y = \sqrt{-x}$ is obtained by reflecting the graph of $y = \sqrt{x}$ through the y axis.



Graph of $y = \sqrt{x}$

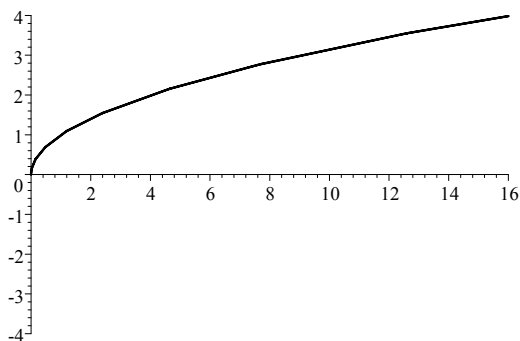


Graph of $y = \sqrt{-x}$

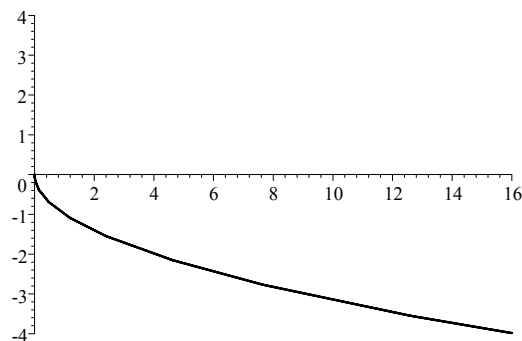
2.2 Reflections Through the x Axis

If we begin with the formula $y = f(x)$ and we replace y with $-y$, then we get the formula $-y = f(x)$ which can also be written as $y = -f(x)$. The graph of $y = -f(x)$ is obtained by reflecting the graph of $y = f(x)$ through the x axis. If the point (x, y) is on the graph of $y = f(x)$, then the point $(x, -y)$ is on the graph of $y = -f(x)$.

Example 8 Starting with the formula $y = \sqrt{x}$, if we replace y with $-y$, then we get the formula $y = -\sqrt{x}$. The graph of $y = -\sqrt{x}$ is obtained by reflecting the graph of $y = \sqrt{x}$ through the x axis.



Graph of $y = \sqrt{x}$



Graph of $y = -\sqrt{x}$

Exercise 9 Perform the following transformations to the function $y = x^2$.

1. a reflection through the y axis.
2. a reflection through the x axis..

In each case, write the formula that gives the requested transformation and draw the graph of the transformed function.

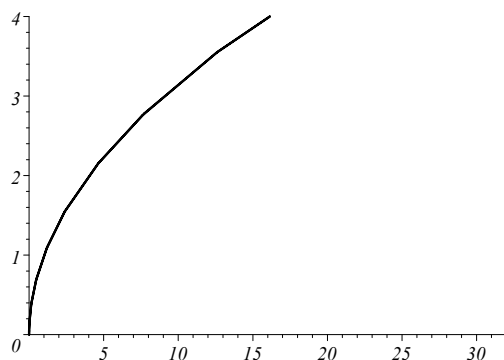
Exercise 10 Repeat the above exercise for the functions $y = -2x + 1$, $y = |x|$, and $y = 1/x$.

3 Expansions and Compressions

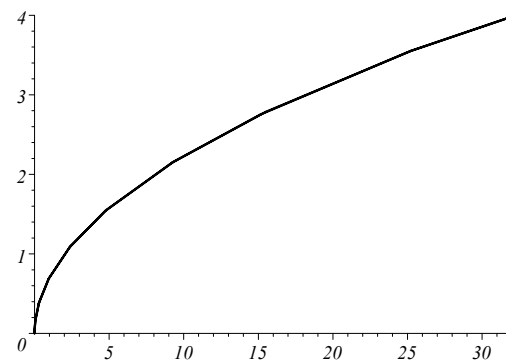
3.1 Horizontal Expansions and Compressions

If we begin with the formula $y = f(x)$ and we replace x with kx where k is a positive constant, then we get the formula $y = f(kx)$. The graph of $y = f(kx)$ is obtained by expanding (or compressing) the graph of $y = f(x)$ horizontally by a factor of $1/k$. Note that if $0 < k < 1$, then an expansion is performed; whereas if $k > 1$, then a compression is performed. If the point (x, y) is on the graph of $y = f(x)$, then the point $(\frac{1}{k}x, y)$ is on the graph of $y = f(kx)$.

Example 11 Starting with the formula $y = \sqrt{x}$, if we replace x with $\frac{1}{2}x$, then we get the formula $y = \sqrt{\frac{1}{2}x}$. The graph of $y = \sqrt{\frac{1}{2}x}$ is obtained by expanding the graph of $y = \sqrt{x}$ horizontally by a factor of 2.

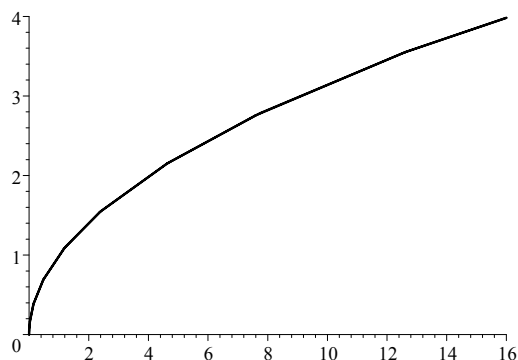


Graph of $y = \sqrt{x}$

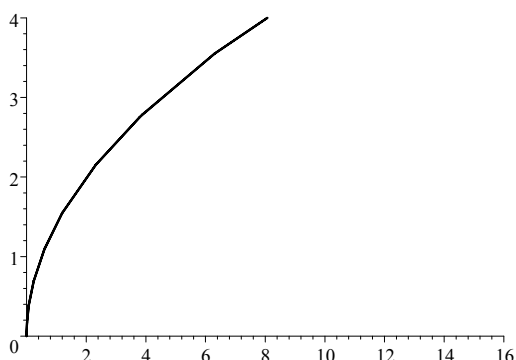


Graph of $y = \sqrt{\frac{1}{2}x}$

Example 12 Starting with the formula $y = \sqrt{x}$, if we replace x with $2x$, then we get the formula $y = \sqrt{2x}$. The graph of $y = \sqrt{2x}$ is obtained by compressing the graph of $y = \sqrt{x}$ horizontally by a factor of $1/2$.



Graph of $y = \sqrt{x}$

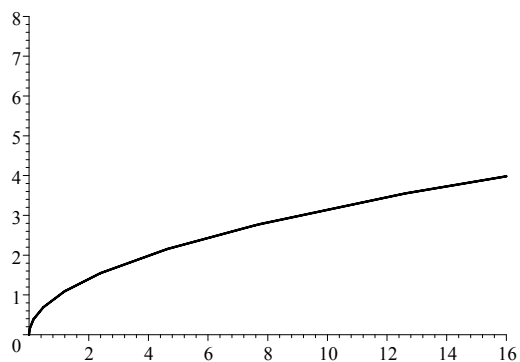


Graph of $y = \sqrt{2x}$

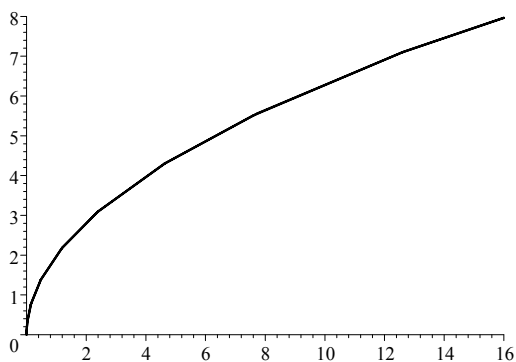
3.2 Vertical Expansions and Compressions

If we begin with the formula $y = f(x)$ and we replace y with ky where k is a positive constant, then we get the formula $ky = f(x)$ which can also be written as $y = \frac{1}{k}f(x)$. The graph of $y = \frac{1}{k}f(x)$ is obtained by expanding (or compressing) the graph of $y = f(x)$ vertically by a factor of $1/k$. Note that if $0 < k < 1$, then an expansion is performed; whereas if $k > 1$, then a compression is performed. If the point (x, y) is on the graph of $y = f(x)$, then the point $(x, \frac{1}{k}y)$ is on the graph of $y = \frac{1}{k}f(x)$.

Example 13 Starting with the formula $y = \sqrt{x}$, if we replace y with $\frac{1}{2}y$, then we get the formula $\frac{1}{2}y = \sqrt{x}$ which can also be written as $y = 2\sqrt{x}$. The graph of $y = 2\sqrt{x}$ is obtained by expanding the graph of $y = \sqrt{x}$ vertically by a factor of 2.

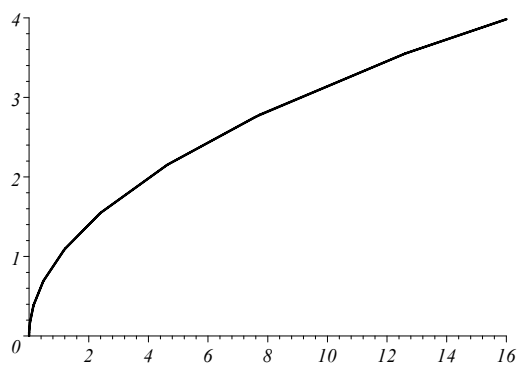


Graph of $y = \sqrt{x}$

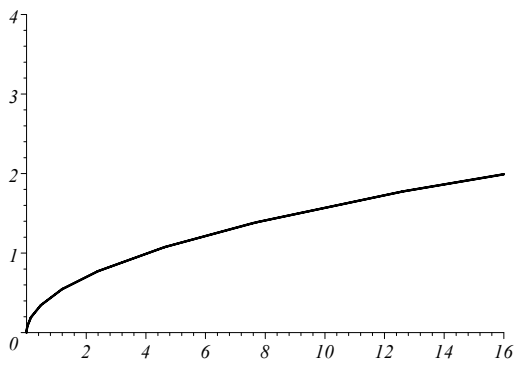


Graph of $y = 2\sqrt{x}$

Example 14 Starting with the formula $y = \sqrt{x}$, if we replace y with $2y$, then we get the formula $2y = \sqrt{x}$ which can also be written as $y = \frac{1}{2}\sqrt{x}$. The graph of $y = \frac{1}{2}\sqrt{x}$ is obtained by compressing the graph of $y = \sqrt{x}$ vertically by a factor of $1/2$.



Graph of $y = \sqrt{x}$



Graph of $y = \frac{1}{2}\sqrt{x}$

Exercise 15 Perform the following transformations to the function $y = x^2$.

1. a horizontal expansion by a factor of 4.
2. a horizontal compression by a factor of $1/4$.

3. a vertical expansion by a factor of 4.
4. a vertical compression by a factor of $1/4$.

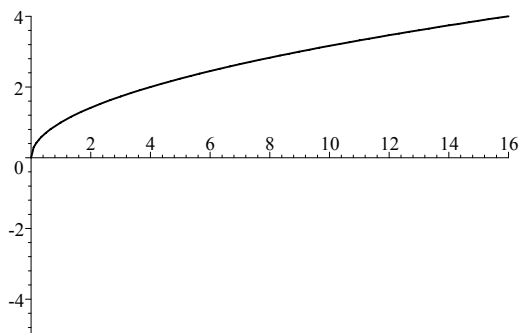
In each case, write the formula that gives the requested transformation and draw the graph of the transformed function

Exercise 16 Repeat the above exercise for the functions $y = -2x + 1$, $y = |x|$, and $y = 1/x$.

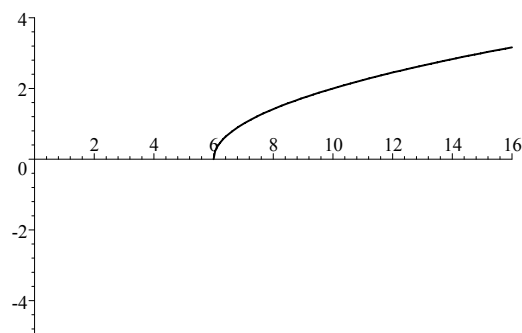
4 Performing a Sequence of Transformations

Starting with a “basic” function such as $y = \sqrt{x}$, we can perform a sequence of transformations to obtain the graph of a similar but “less basic” function. As an example, let us perform a sequence of transformations that lead to the graph of $y = \sqrt{4x - 6} - 5$.

Step 1. We begin with $y = \sqrt{x}$ and replace x with $x - 6$ to obtain $y = \sqrt{x - 6}$. The graph of $y = \sqrt{x - 6}$ is a translation of the graph of $y = \sqrt{x}$ to the right by 6 units.

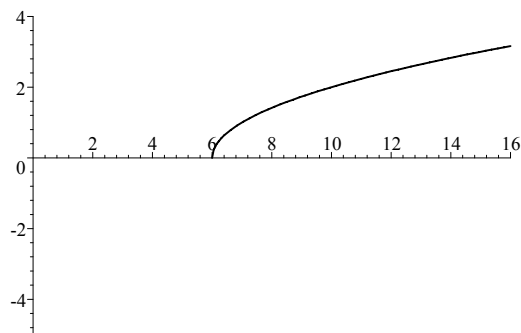


Graph of $y = \sqrt{x}$

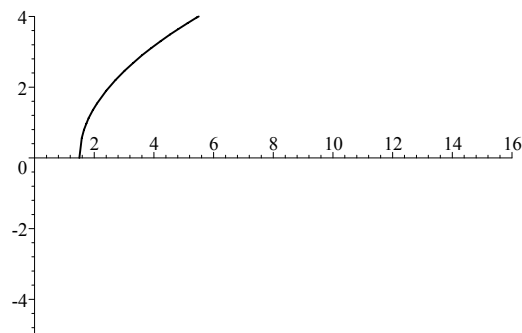


Graph of $y = \sqrt{x - 6}$

Step 2. We begin with $y = \sqrt{x - 6}$ and replace x with $4x$ to obtain $y = \sqrt{4x - 6}$. The graph of $y = \sqrt{4x - 6}$ is a horizontal compression of the graph of $y = \sqrt{x - 6}$ by a factor of $1/4$.

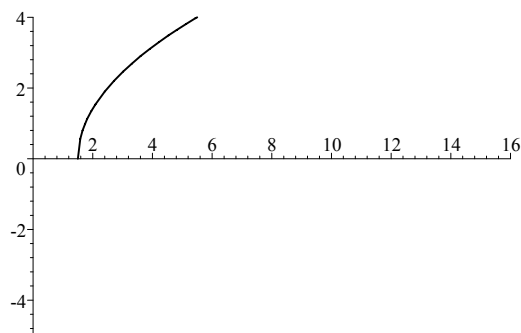


Graph of $y = \sqrt{x - 6}$

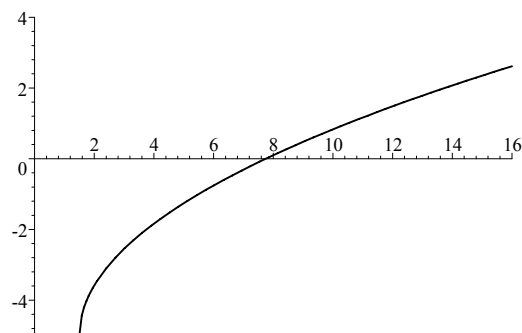


Graph of $y = \sqrt{4x - 6}$

Step 3. We begin with $y = \sqrt{4x - 6}$ and replace y with $y + 5$ to obtain $y + 5 = \sqrt{4x - 6}$ which is the same as $y = \sqrt{4x - 6} - 5$. The graph of $y = \sqrt{4x - 6} - 5$ is a translation of the graph of $y = \sqrt{4x - 6}$ downward by 5 units.



Graph of $y = \sqrt{4x - 6}$



Graph of $y = \sqrt{4x - 6} - 5$

Exercise 17 *In the exercises that follow, fill in the blanks. The first exercise is done as an example.*

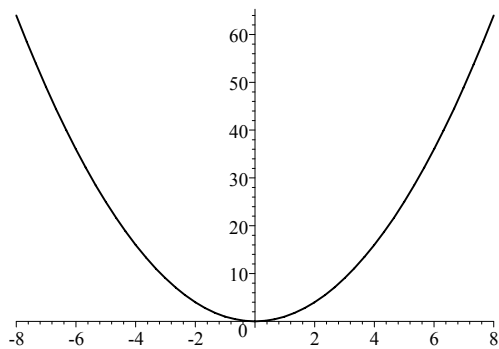
1. To obtain the graph of $y = |x + 3|$, we translate the graph of $y = |x|$ to the left by 3 units.

2. To obtain the graph of $y = -|x|$, we _____ the graph of $y = |x|$
_____.
3. To obtain the graph of $y = -x + 5$, we _____ the graph of
 $y = x + 5$ _____.
4. To obtain the graph of _____, we expand the graph of $y = x + 5$
vertically by a factor of 3.
5. To obtain the graph of $y = 4x^3 - 6$, we translate the graph of _____
downward by 6 units.
6. To obtain the graph of $y = \frac{1}{x+5}$, we _____ the graph of $y = \frac{1}{x}$
_____.
7. To obtain the graph of $y = 4x^2$, we compress the graph of _____
horizontally by a factor of $1/2$.
8. To obtain the graph of $y = 4x^2$, we expand the graph of _____
vertically by a factor of 4.
9. To obtain the graph of _____, we translate the graph of $y = x^3$
to the left by a 8 units.
10. To obtain the graph of $y = |6x|$, we _____ the graph of $y = |x|$
_____.
11. To obtain the graph of $y = -x^2$, we _____ the graph of $y = x^2$
_____.

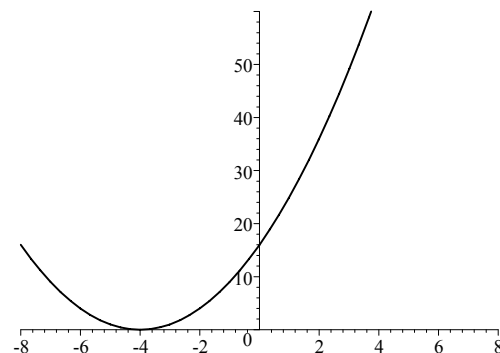
Exercise 18 Perform a sequence of transformations to obtain the graph of each of the following functions. Do this step by step and explain each step along the way. The first step of the first exercise is done as an example.

1. $y = (x + 4)^2 - 7$

Step 1. We begin with $y = x^2$ and replace x with $x + 4$ to obtain $y = (x + 4)^2$. The graph of $y = (x + 4)^2$ is a translation of the graph of $y = x^2$ to the left by 4 units.



Graph of $y = x^2$



Graph of $y = (x + 4)^2$

2. $y = 5x - 10$
3. $y = \sqrt{3x + 9} + 5$
4. $y = 8 - \sqrt{10 - x}$
5. $y = 2(x + 4)^3$
6. $y = \frac{5}{x}$
7. $y = -4|-2x + 3|$