1 The Chain Rule Answers to Exercises

1. (a) The derivative of $z = \sin^3 x$ is

$$\frac{dz}{dx} = 3\sin^2 x \cos x.$$

(b) The derivative of $z = \sin\left(x^3\right)$ is

$$\frac{dz}{dx} = \cos\left(x^3\right) \cdot 3x^2.$$

(c) The derivative of $z = \left(4x^5 - 5x^4 + 9x^3 + x^2 - 2x - 12\right)^4$ is

$$\frac{dz}{dx} = 4\left(4x^5 - 5x^4 + 9x^3 + x^2 - 2x - 12\right)^3\left(20x^4 - 20x^3 + 27x^2 + 2x - 2\right).$$

(d) The derivative of $f(t) = \left(4t^4 + \cos t + 2\right)^3$ is

$$f'(t) = 3\left(4t^4 + \cos t + 2\right)^2\left(16t^3 - \sin t\right).$$

(e) The derivative of $g(\theta) = \tan^2\theta$ is

$$\begin{aligned} g'(\theta) &= 2\tan\theta \cdot \sec\theta\tan\theta \\ &= 2\sec\theta\tan^2\theta. \end{aligned}$$

(f) The derivative of

$$y = \frac{\sin t + 1}{\cos t + 2}$$

is

$$\frac{dy}{dt} = \frac{(\cos t + 2)(\cos t) - (\sin t + 1)(-\sin t)}{(\cos t + 2)^2}.$$

(g) The derivative of

$$y = \frac{\left(\cos^2 x + x\right)^3}{x^4 + 3x^2 + 1}$$

is

$$\frac{dy}{dx} = \frac{\left(x^4 + 3x^2 + 1\right) \cdot 3\left(\cos^2 x + x\right)^2 \cdot (-2\sin x\cos x + 1) - \left(\cos^2 x + x\right)^3\left(4x^3 + 6x\right)}{\left(x^4 + 3x^2 + 1\right)^2}$$

3. We are given that the tank is cone shaped and has height h ft. and radius r ft. (at the top). We are also given that the tank is being lled at c ft^3/min.

The volume of water in the tank is

$$V = \frac{1}{3}\pi R^2 D$$

(where D is the depth of the water and R is the radius of the water surface). Using the similar triangle relationship, we have

$$\frac{R}{D} = \frac{r}{h}$$

which gives us

$$R = \frac{r}{h} D.$$

We can now write V as a function of D only:

$$V = \frac{1}{3}\pi \left(\frac{r}{h} D\right)^2 D$$

which simpli es to

$$V = \frac{\pi r^2}{3h^2} D^3.$$

This gives us

$$\frac{dV}{dD} = \frac{\pi r^2}{h^2} D^2.$$

Since the tank is lling at rate c ft^3/min, we also have

$$V = ct$$

which gives us

$$\frac{dV}{dt} = c.$$

By the Chain Rule,

$$\begin{aligned} \frac{dD}{dt} &= \frac{\frac{dV}{dt}}{\frac{dV}{dD}} \\ &= \frac{c}{\frac{\pi r^2}{h^2} D^2} \\ &= \frac{ch^2}{\pi r^2 D^2} \\ &= \frac{\frac{ch^2}{\pi r^2}}{D^2}. \end{aligned}$$

This shows that

$$\frac{dD}{dt} = \frac{a}{D^2}$$

where

$$a = \frac{ch^2}{\pi r^2}.$$

When the water is halfway to the top, we have

$$D = \frac{1}{2}h.$$

Evaluating dD/dt at this value of D gives us

$$\begin{aligned} \left.\frac{dD}{dt}\right|_{D=\frac{1}{2}h} &= \frac{a}{\left(\frac{1}{2}h\right)^2} \\ &= \frac{4a}{h^2} \\ &= \frac{4\left(\frac{ch^2}{\pi r^2}\right)}{h^2} \\ &= \frac{4c}{\pi r^2} \end{aligned}$$

so the water is rising at rate $4c/\pi r^2$ ft/min when the tank is half full. It is interesting to note that this rate does not depend on h (the height of the tank).

5. The derivative of $z = \left(x^2 + 3x - 12\right)^6$ is

$$\frac{dz}{dx} = 6\left(x^2 + 3x - 12\right)^5 (2x + 3).$$

The local maxima and minima occur at points x where $dz/dx = 0$. Setting

$$6\left(x^2 + 3x - 12\right)^5 (2x + 3) = 0,$$

we obtain

$$x^2 + 3x - 12 = 0$$

, Solution is: $\left\{x = -\frac{3}{2} + \frac{1}{2}\sqrt{57}\right\}, \left\{x = -\frac{3}{2} - \frac{1}{2}\sqrt{57}\right\}$or

$$2x + 3 = 0.$$

Using the Quadratic formula, we see that the solutions of $x^2 + 3x - 12 = 0$ are

$$x = \frac{-3 + \sqrt{57}}{2} \approx 2.2749$$

and

$$x = \frac{-3 - \sqrt{57}}{2} \approx -5.2749.$$

The solution of $2x + 3 = 0$ is

$$x = -\frac{3}{2} = -1.5.$$

Comparing these ndings with the graph in Figure 3 (on page 10 of the notes), we conclude that z has local minima occurring at $x \approx -5.2749$ and at $x \approx 2.2749$ and that f has a local maximum occurring at $x = -1.5$. These are the only local extrema of z because they are the only values of x where $dz/dx = 0$.

5. For the function $g(x) = \left(2x^2 - 5x - 3\right)^3$, we have

$$g'(x) = 3\left(2x^2 - 5x - 3\right)^2 (4x - 5)$$

and (using the Product Rule and Chain Rule),

$$g''(x) = 3\left(2x^2 - 5x - 3\right)^2 (4) + 6\left(2x^2 - 5x - 3\right)(4x - 5)(4x - 5).$$

To determine intervals on which f is increasing/decreasing, we rst look for points where $g'(x) = 0$. Setting

$$3\left(2x^2 - 5x - 3\right)^2 (4x - 5) = 0$$

gives

$$2x^2 - 5x - 3 = 0$$

or

$$4x - 5 = 0.$$

The solutions of $2x^2 - 5x - 3 = 0$ are $x = -1/2$ and $x = 3$ and the solution of $4x - 5 = 0$ is $x = 5/4$. Thus the critical points are $x = -1/2$, $x = 5/4$, and $x = 3$.

Choosing a value of x less than $-1/2$ (let s take $x = -1$) we get

$$g'(-1) = 3\left(2(-1)^2 - 5(-1) - 3\right)^2 (4(-1) - 5) = -432.$$

Since $g'(-1) < 0$, we conclude that g is decreasing on the interval $(-\infty, -1/2)$.

Choosing $x = 0$ (which is between $-1/2$ and $5/4$), we get

$$g'(0) = 3\left(2(0)^2 - 5(0) - 3\right)^2 (4(0) - 5) = -135.$$

Since $g'(0) < 0$, we conclude that g is decreasing on the interval $(-1/2, 5/4)$.

Choosing $x = 2$ (which is between $5/4$ and 3), we get

$$g'(2) = 3\left(2(2)^2 - 5(2) - 3\right)^2 (4(2) - 5) = 225.$$

Since $g'(2) > 0$, we conclude that g is increasing on the interval $(5/4, 3)$.

Choosing $x = 4$ (which is greater than 3), we get

$$g'(4) = 3\left(2(4)^2 - 5(4) - 3\right)^2 (4(4) - 5) = 2,673.$$

Since $g'(4) > 0$, we conclude that g is increasing on the interval $(3, \infty)$.

In summary, g is decreasing on the interval $(-\infty, 5/4)$ and increasing on the interval $(5/4, \infty)$. This means that g has a local minimum at $x = 5/4$. The local minimum value of g at $x = 5/4$ is

$$g(5/4) = \left(2(5/4)^2 - 5(5/4) - 3\right)^3 \approx -230.$$

This is in fact the **absolute** minimum value of g.

To determine the concavity of g, we rst look for points x at which $g''(x) = 0$. Recall that

$$g''(x) = 2\left(2x^2 - 5x - 3\right) + (4x - 5)(4x - 5)$$

which can be factored as

$$g''(x) = 6\left(2x^2 - 5x - 3\right)\left(2\left(2x^2 - 5x - 3\right) + (4x - 5)(4x - 5)\right).$$

After more simpli cation, we obtain

$$g''(x) = 6\left(2x^2 - 5x - 3\right)\left(20x^2 - 50x + 19\right).$$

Setting $g''(x) = 0$ gives

$$2x^2 - 5x - 3 = 0$$

or

$$20x^2 - 50x + 19 = 0.$$

The solutions of the rst of these equations are $x = -1/2$ and $x = 3$ and the solutions of the second equation are

$$x = \frac{25 + 7\sqrt{5}}{20} \approx 2.03$$

and

$$x = \frac{25 - 7\sqrt{5}}{20} \approx 0.47.$$

We now divide the x axis into ve intervals, $(-\infty, -1/2)$, $(-1/2, 0.47)$, $(0.47, 2.03)$, $(2.03, 3)$, and $(3, \infty)$, and pick a point in each of these intervals at which to evaluate g''. We obtain

$$\begin{aligned} g''(-1) &= 2,136 > 0 \\ g''(0) &= -342 < 0 \\ g''(1) &= 396 > 0 \\ g''(2.5) &= -342 < 0 \\ g''(4) &= 7,506 > 0 \end{aligned}$$

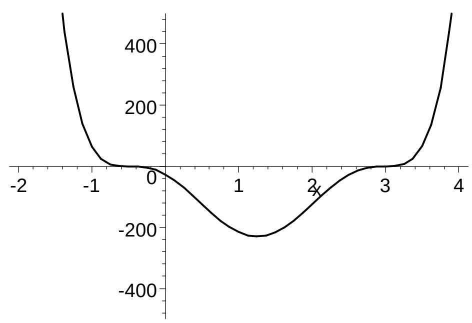
We deduce that g is concave up on the intervals

$$\left(-\infty, -\frac{1}{2}\right), \quad \left(\frac{25-7\sqrt{5}}{20}, \frac{25+7\sqrt{5}}{20}\right), \text{ and } (3, \infty)$$

and that g is concave down on the intervals

$$\left(-\frac{1}{2}, \frac{25-7\sqrt{5}}{20}\right) \quad \text{and} \quad \left(\frac{25+7\sqrt{5}}{20}, 3\right).$$

The graph of g shown below seems to support all of our ndings.



Graph of $g(x) = \left(2x^2 - 5x - 3\right)^3$

9. The derivative of $y = \sqrt{\sin(t^2+4)} = \left(\sin\left(t^2+4\right)\right)^{1/2}$ is

$$\frac{dy}{dt} = \frac{1}{2}\left(\sin\left(t^2+4\right)\right)^{-1/2} \cdot \cos\left(t^2+4\right) \cdot 2t$$

which can be written as

$$\frac{dy}{dt} = \frac{t\cos\left(t^2+4\right)}{\sqrt{\sin(t^2+4)}.}$$