The Limit Laws

Suppose that

$$\lim_{x\to a} f(x)$$

and

$$\lim_{x\to a} g(x)$$

both exist (meaning that they are real numbers and not ∞ or $-\infty$). Then the following *limit laws* are valid:

1. $\lim_{x\to a}(f(x)+g(x)) = \lim_{x\to a} f(x) + \lim_{x\to a} g(x)$
2. $\lim_{x\to a}(f(x)-g(x)) = \lim_{x\to a} f(x) - \lim_{x\to a} g(x)$
3. If c is a constant, then $\lim_{x\to a}(c \cdot f(x)) = c \cdot \lim_{x\to a} f(x)$
4. $\lim_{x\to a}(f(x)\cdot g(x)) = (\lim_{x\to a} f(x)) \cdot (\lim_{x\to a} g(x))$
5. If $\lim_{x\to a} g(x) \neq 0$, then $\lim_{x\to a}\left(\frac{f(x)}{g(x)}\right) = \frac{\lim_{x\to a} f(x)}{\lim_{x\to a} g(x)}$.

Example *The graphs of two functions, f and g, are shown in Figure* 1 *on page* 111 *of the Stewart textbook. Use these graphs to evaluate the limits:*

$$\lim_{x\to -2}(f(x)+5g(x)) = \_\_\_\_\_\_\_\_\_\_\_\_$$

$$\lim_{x\to 1}(f(x)g(x)) = \_\_\_\_\_\_\_\_\_\_\_\_$$

$$\lim_{x\to 2}\frac{f(x)}{g(x)} = \_\_\_\_\_\_\_\_\_\_\_\_$$

Some Additional Limit Laws

6. If n is a positive integer, then $\lim_{x\to a}((f(x))^n) = (\lim_{x\to a} f(x))^n$
7. If c is a constant, then $\lim_{x\to a} c = c$
8. $\lim_{x\to a} x = a$
9. If n is a positive integer, then $\lim_{x\to a}(x^n) = a^n$
10. If n is a positive integer, then $\lim_{x\to a} \sqrt[n]{x} = \sqrt[n]{a}$ (If n is even, then this is valid only if $a > 0$.)
11. If n is a positive integer, then $\lim_{x\to a} \sqrt[n]{f(x)} = \sqrt[n]{\lim_{x\to a} f(x)}$ (If n is even, then this is valid only if $\lim_{x\to a} f(x) > 0$.)

The "Direct Substitution" Property for Polynomials and Rational Functions

If f is a polynomial or a rational function (meaning a ratio of two polynomials) and a is in the domain of f, then

$$\lim_{x \to a} f(x) = f(a).$$

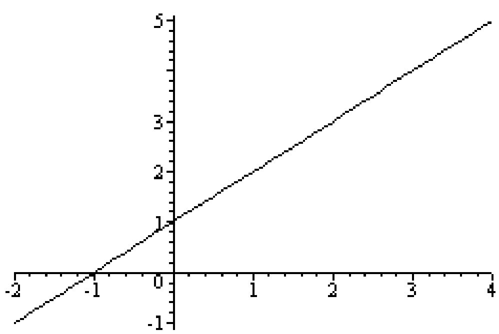
There are in fact many other functions that have this "direct substitution" property. These include most of the functions studied in Precalculus, such as exponential and trigonometric functions. This will be discussed further in Section 2.4.

Example *Find the value of*

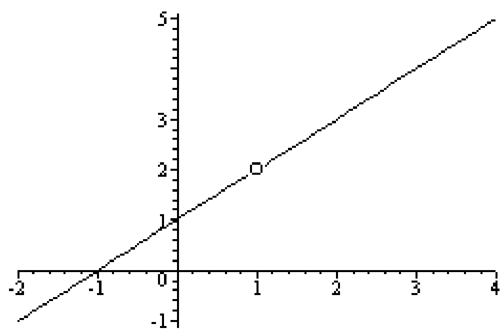
$$\lim_{x \to 1} \frac{x^2 - 1}{x - 1}.$$

Hint: *The key fact to note here is that*

$$\frac{x^2 - 1}{x - 1} = x + 1 \quad \textit{for all } x \neq 1.$$



graph of $y = x + 1$

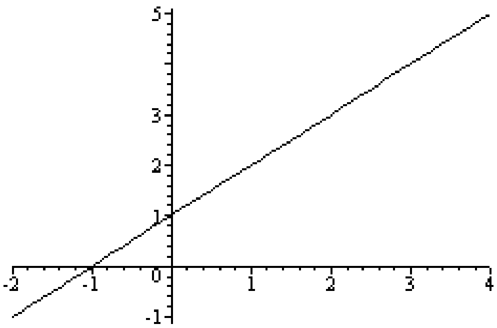


graph of $y = \frac{x^2-1}{x-1}$

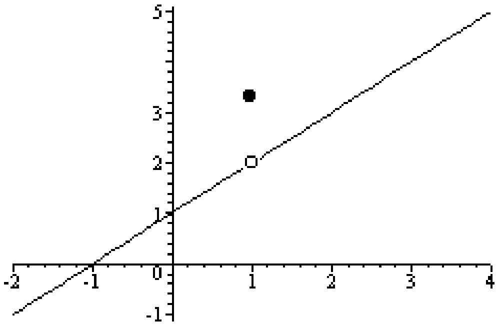
Example *Let g be the function defined by*

$$g(x) = \begin{cases} x + 1 & \textit{if } \; x \neq 1 \\ \pi & \textit{if } \; x = 1 \end{cases}.$$

Find $\lim_{x \to 1} g(x)$.



graph of $y = x + 1$



graph of $y = g(x)$

Example *Evaluate*

$$\lim_{h \to 0} \frac{(3+h)^2 - 9}{h}.$$

Example *Evaluate*

$$\lim_{t \to 0} \frac{\sqrt{t^2+9} - 3}{t^2}.$$

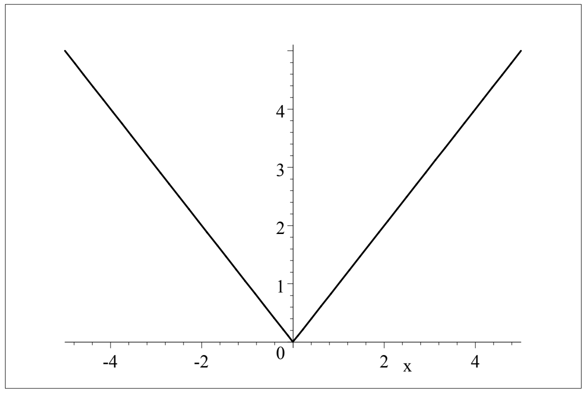
Recall

$$\lim_{x \to a} f(x) = L \quad \text{if and only if} \quad \lim_{x \to a^-} f(x) = \lim_{x \to a^+} f(x) = L.$$

Example *Show that* $\lim_{x \to 0} |x| = 0$ *by computing the left and right–hand limits of this function at* 0.

Hint:

$$|x| = \begin{cases} x & \text{if } x \geq 0 \\ -x & \text{if } x < 0 \end{cases}.$$



Graph of $y = |x|$

Example *Explain why*

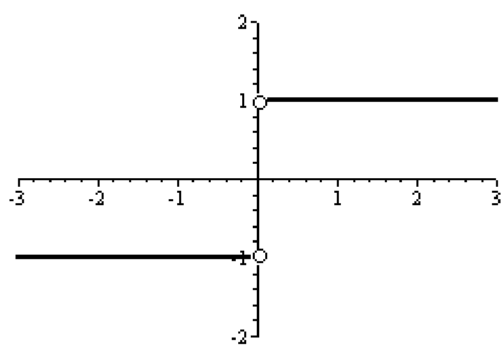
$$\lim_{x \to 0} \frac{|x|}{x} \quad \text{does not exist}$$

by computing the left and right–hand limits of this function at 0.

Hint:

$$\frac{|x|}{x} = \begin{cases} 1 & \text{if } x > 0 \\ -1 & \text{if } x < 0 \end{cases}$$

(and this function is not defined at $x = 0$*).*



graph of $y = \frac{|x|}{x}$

Example *The greatest integer function is defined by*

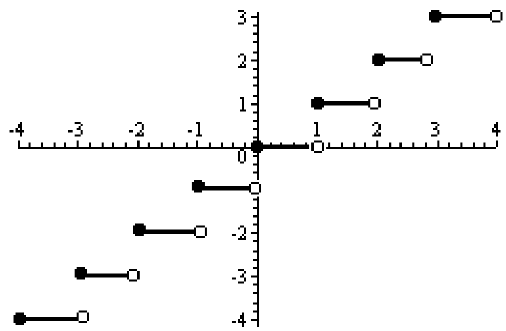
$$\lfloor x \rfloor = \text{the largest integer that is less than or equal to } x.$$

(For example, $\lfloor 2 \rfloor = 2$*,* $\lfloor 5.7 \rfloor = 5$*, and* $\lfloor -3.4 \rfloor = -4$*.)*

Explain why

$$\lim_{x\to 3} \lfloor x \rfloor \quad \text{does not exist.}$$

Hint: *Compute the left and right–hand limits of this function at* 3.



Graph of $y = \lfloor x \rfloor$

The Comparison and Squeeze Theorems for Limits

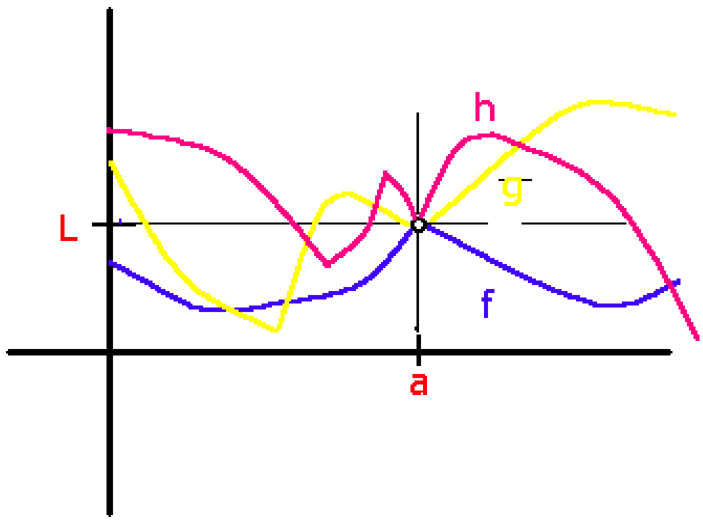
Theorem (The Comparison Theroem) *Suppose that* $f(x) \le g(x)$ *for all x sufficiently near a (but not necessarily at* $x = a$*) and suppose that* $\lim_{x\to a} f(x)$ *and* $\lim_{x\to a} g(x)$ *both exist. Then*

$$\lim_{x\to a} f(x) \le \lim_{x\to a} g(x).$$

Theorem (The Squeeze Theorem) *Suppose that* $f(x) \le g(x) \le h(x)$ *for all x sufficiently near a (but not necessarily at* $x = a$*) and suppose that* $\lim_{x\to a} f(x)$ *and* $\lim_{x\to a} h(x)$ *both exist and are equal. (In other words, suppose that* $\lim_{x\to a} f(x) = \lim_{x\to a} h(x) = L$ *where is a real number). Then*

$$\lim_{x\to a} g(x) = L.$$

Proof The basic idea of the Squeeze Theorem is illustrated in the picture below. A formal proof requires the “advanced calculus” definition of limit (just as do the proof of the limit laws that we stated earlier without proof).



Example *Use the Squeeze Theorem to show that*

$$\lim_{x \to 0} x^2 \sin\left(\frac{1}{x}\right) = 0.$$

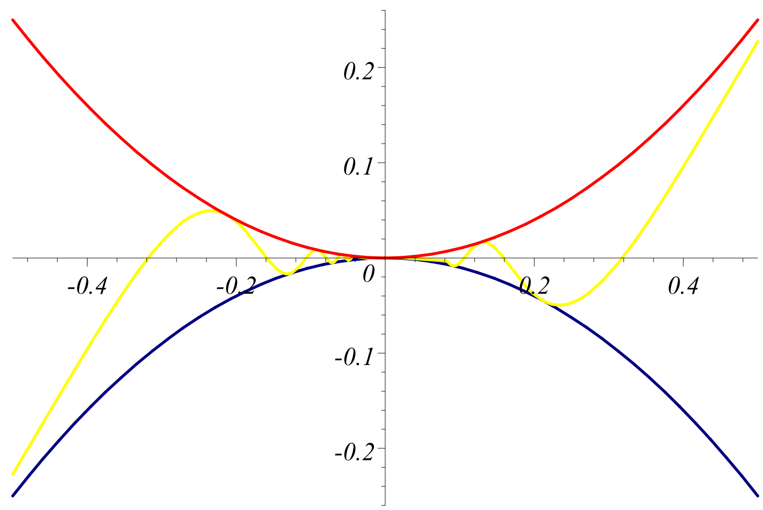
Hint: *Look at the three graphs of*

$$y = -x^2$$

$$y = x^2 \sin\left(\frac{1}{x}\right)$$

$$y = x^2$$

shown below.



0.2
0.1
-0.4
-0.2
0
0.2
0.4
-0.1
-0.2