

# Falling Objects

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When an object falls through the air (sufficiently near the surface of the earth), there are two basic forces at work on the object. One of these forces is the *weight*,  $W$ , of the object which pulls the object toward the earth. The other force is a *drag*,  $D$ , which pushes the object away from the earth. (See Figure 1.)

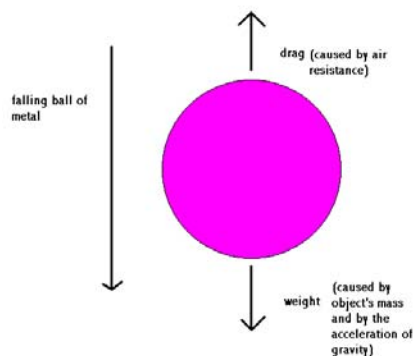


Figure 1: Forces acting on a falling object

The weight of the object is defined to be the mass,  $m$ , of the object times the acceleration of gravity,  $g$ . Thus

$$W = mg. \tag{1}$$

The value of  $g$  is a constant that has been measured experimentally. It is  $g = 9.8 \text{ m/s}^2$ . (Actually, the value of  $g$  decreases as we move farther away

from the earth, but the value  $g = 9.8$  is sufficiently accurate for studying objects falling near the earth). In equation (1), we assume that the mass of the object is measured in kilograms (kg) so that the units of weight are Newtons ( $\text{kg m / s}^2$ ).

The drag force,  $D$ , that acts on the falling object is caused by the resistance of the air. As the object falls, the air beneath it pushes on it and slows down its fall. The faster the object is moving, the harder the air pushes. Thus, the drag force depends on how fast the object is falling. For relatively small objects that are roughly spherical in shape, it is reasonable to simply assume that the drag force is proportional to the velocity and hence that

$$D = bv \tag{2}$$

where  $v$  is the velocity of the falling object measured in units of  $\text{m / s}$  and  $b$  is a constant of proportionality called the *drag coefficient* with units of  $\text{kg / s}$ . The drag force defined by equation (2) is called *viscous drag*. If the falling object that is being modelled is very massive and/or is not roughly spherical in shape, then it is more reasonable to assume that the drag force is proportional to the square of the velocity via the equation

$$D = \frac{1}{2}c\rho Av^2 \tag{3}$$

where  $\rho$  is the air density (with units of  $\text{kg / m}^3$ ),  $A$  is the frontal surface area of the object (with units of  $\text{m}^2$ ), and  $c$  is a drag coefficient that has no units. The type of drag defined by equation (3) is called *Newton drag*. The units of drag, whether we are assuming viscous drag or Newton drag, are Newtons ( $\text{kg m / s}^2$ ).

## 1 A Falling Object With No Drag

Let us begin by attempting to describe the motion of a falling object that does not experience a drag force. An object that falls without experiencing a drag force is called a *free-falling* object. Free-fall can occur only if the object is falling in a *vacuum* (meaning that there are no air molecules present to slow down the fall). Such conditions do not occur in the “real world” but can be simulated in a laboratory.

First, we will agree to measure the falling object’s displacement from the earth with a variable,  $y$ , and we will assign  $y = 0$  to be the surface of the earth. (See Figure 2.)

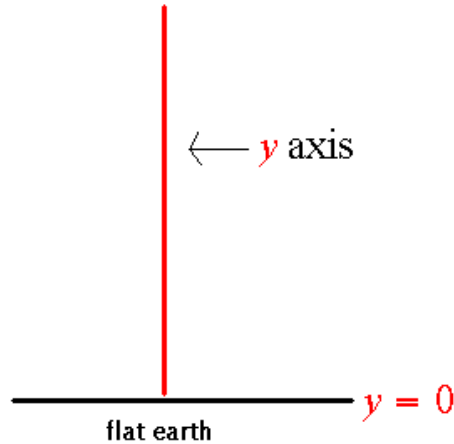


Figure 2: Fall takes place along the  $y$  axis.

Next, we recall that velocity,  $v$ , is the rate of change of the displacement,  $y$ , and that acceleration,  $a$ , is the rate of change of velocity. Thus we have

$y$  = displacement (with units m)

$v = \frac{dy}{dt}$  = velocity (with units m/s)

$a = \frac{dv}{dt} = \frac{d^2y}{dt^2}$  = acceleration (with units of  $\text{m/s}^2$ ).

Note that, for a falling object, we always have  $y > 0$  (because the object is above the surface of the earth),  $v < 0$  (because  $v = dy/dt$  and  $y$  is decreasing), and  $a < 0$  (because  $a = dv/dt$  and  $v$  is decreasing). The fact that  $a < 0$  is a subtle point: The free-falling object speeds up as it falls, meaning that the **speed** of the object increases as it falls. However, speed is defined to be the absolute value of velocity. Hence, an increase in speed corresponds to a decrease in velocity. The velocity becomes more and more negative as the object falls.

Using Newton's Second Law of Motion which states (in simplified form) that *force equals mass times acceleration*, and using our assumption that the only force acting on the free-falling object is its weight,  $W = mg$ , and

that this force is exerted in the negative (downward) direction, we obtain the equation

$$ma = -mg$$

or simply

$$a = -g.$$

(Recall that  $g = 9.8 \text{ m/s}^2$  is a constant.) Since  $a = dv/dt$ , we obtain the differential equation

$$\frac{dv}{dt} = -g$$

which has general solution

$$v = -gt + C.$$

Assuming that the initial velocity of the object (at time  $t = 0$ ) is  $v(0) = v_0 \leq 0$ , we obtain the particular solution

$$v = -gt + v_0.$$

Since  $v = dy/dt$ , we obtain the differential equation

$$\frac{dy}{dt} = -gt + v_0$$

which has general solution

$$y = -\frac{1}{2}gt^2 + v_0t + C.$$

Finally, if we assume that the initial displacement of the object (at time  $t = 0$ ) is  $y(0) = y_0$ , then we obtain the particular solution

$$y = -\frac{1}{2}gt^2 + v_0t + y_0.$$

In summary, we have found that a free-falling object that begins at initial displacement  $y(0) = y_0$  with initial velocity  $v(0) = v_0$  has velocity

$$v = -gt + v_0 \tag{4}$$

and displacement

$$y = -\frac{1}{2}gt^2 + v_0t + y_0. \tag{5}$$

The velocity function (4) and displacement function (5) remain in effect over the time interval  $0 \leq t \leq T$  where  $T$  is the time that it takes for the object to reach the ground. Note that the velocity and displacement functions **do not depend** on the mass,  $m$ , of the object!

## 2 A Falling Object With Viscous Drag Force

If we assume that our falling object experiences a viscous drag force,

$$D = bv,$$

then we obtain

$$ma = -mg - bv.$$

Note that  $-mg < 0$  and that  $-bv > 0$  (because  $b > 0$  and  $v < 0$ ). In other words, the drag force is in the opposite direction of the force of gravity.

Dividing both sides of the above equation by  $m$ , we obtain

$$a = -g - \frac{b}{m}v.$$

Defining a new parameter,  $p$ , by  $p = b/m$ , we obtain the differential equation

$$\frac{dv}{dt} = -g - pv.$$

We can solve this differential equation by separating variables: Writing the equation as

$$\int \frac{dv}{pv + g} = - \int dt$$

and making the simple substitution  $u = pv + g$ ,  $du = p dv$ , we obtain

$$\frac{1}{p} \int \frac{1}{u} du = - \int dt$$

which gives us

$$\frac{1}{p} \ln |u| = -t + C$$

or

$$\ln |pv + g| = -pt + C$$

or

$$|pv + g| = Ce^{-pt}$$

or

$$pv + g = Ce^{-pt}$$

or

$$v = -\frac{g}{p} + Ce^{-pt}.$$

Assuming an initial velocity of  $v(0) = v_0$ , we obtain

$$v_0 = -\frac{g}{p} + C$$

or

$$C = v_0 + \frac{g}{p}.$$

Thus the velocity function for the falling object is

$$v = -\frac{g}{p} + \left(v_0 + \frac{g}{p}\right) e^{-pt}.$$

Next, using the fact that  $dy/dt = v$ , we obtain the differential equation

$$\frac{dy}{dt} = -\frac{g}{p} + \left(v_0 + \frac{g}{p}\right) e^{-pt}$$

which has general solution

$$y = -\frac{g}{p}t - \frac{1}{p} \left(v_0 + \frac{g}{p}\right) e^{-pt} + C.$$

Using the initial condition  $y(0) = y_0$ , we obtain

$$y_0 = -\frac{1}{p} \left(v_0 + \frac{g}{p}\right) + C$$

or

$$C = y_0 + \frac{1}{p} \left(v_0 + \frac{g}{p}\right).$$

This gives us the displacement function

$$y = -\frac{g}{p}t - \frac{1}{p} \left(v_0 + \frac{g}{p}\right) e^{-pt} + y_0 + \frac{1}{p} \left(v_0 + \frac{g}{p}\right)$$

which can also be written as

$$y = -\frac{g}{p}t + \frac{1}{p} \left(v_0 + \frac{g}{p}\right) (1 - e^{-pt}) + y_0.$$

Now, making the substitution  $p = b/m$  and using the fact that  $W = mg$ , we conclude that the velocity and displacement functions for a falling object that experiences a viscous drag force are

$$v = -\frac{W}{b} + \left(v_0 + \frac{W}{b}\right) e^{-(b/m)t} \quad (6)$$

and

$$y = -\frac{W}{b}t + \frac{m}{b} \left(v_0 + \frac{W}{b}\right) (1 - e^{-(b/m)t}) + y_0. \quad (7)$$

The velocity function (6) and displacement function (7) remain in effect over the time interval  $0 \leq t \leq T$  where  $T$  is the time that it takes for the object to reach the ground. Also, we observe that the velocity and displacement of the object **do depend** on the mass,  $m$ , of the object. In particular, since  $\lim_{t \rightarrow \infty} e^{-(b/m)t} = 0$ , we see that if the object falls for a very long time, the for large values of  $t$  we have

$$v \approx -\frac{W}{b}.$$

This means that the velocity of the object becomes roughly constant as it continues to fall. The quantity  $W/b$  is called the *terminal velocity* of the falling object. Whereas a free-falling object continues to speed up, a falling object that experiences drag asymptotically approaches a constant velocity that depends on the weight of the object and on the drag coefficient.

### 3 Falling Object with Newton Drag Force

To model a falling object that experiences a Newton drag force,

$$D = \frac{1}{2}c\rho Av^2,$$

we use the differential equation

$$\frac{dv}{dt} = -g + \frac{1}{2m}c\rho Av^2.$$

This differential equation is separable and can be solved explicitly for  $v$ . (A partial fraction decomposition is involved.) However, once  $v$  has been found, it is very difficult to solve the differential equation

$$\frac{dy}{dt} = v$$

in order to obtain an explicit formula for the displacement function  $y$ . Nonetheless, the velocity function can be analyzed to learn that, as expected, the falling object approaches a terminal velocity. In this case, the terminal velocity is found to depend on the weight,  $W$ , the drag coefficient,  $c$ , the air density,  $\rho$ , and the surface area,  $A$ , of the falling object.