Continuous and Discrete Integrals

The Purpose of this Document

This document discusses some facts about Riemann-Stieltjes integrals of the form

\[ \int_a^b f d\phi \]

that follow from the fact that an increasing function \( \phi \) can always be decomposed into the sum of two increasing functions, one of which increases only by jumping and the other is continuous. This discussion will provide us with some further insight into the notion of integrability of a given bounded function \( f \) with respect to a given increasing function \( \phi \).

Then we shall turn our attention to integrals in which both \( f \) and \( \phi \) are increasing functions on the interval \( [a, b] \) and we shall see that if \( f \) and \( \phi \) are increasing on an interval \( [a, b] \) then the integral \( \int_a^b f d\phi \) always exists and, of course, so does the integral \( \int_a^b \phi df \). The relationship between these two integrals is the integration by parts formula.

In order to read this document successfully you should be familiar with the concept of countability of sets that appears in the alternative chapter on set theory and you should have read at least some of the optional material on rearrangements of infinite series.

Continuous and Discrete Increasing Functions

As we know, an increasing function \( \phi \) is continuous at a given number \( x \) if and only if the jump \( J(\phi, x) \) of \( \phi \) at \( x \) is zero. If an increasing function is continuous then it has no jumps, and it “varies continuously” on each interval without actually changing at any one point.

In our motivation of the Riemann-Stieltjes integral we pictured an increasing function \( \phi \) as being the distribution of mass in a wire and, according to this viewpoint, a function \( \phi \) is continuous when the wire has no beads on it. At the opposite extreme we might consider a wire whose mass lies only in its beads. In such a case, the function \( \phi \) would increase only by jumping and have no continuous character at all. A function of this type is said to vary discretely and is called a discrete function. After giving a precise definition of a discrete function, we shall show that every increasing function can be written as the sum of a continuous function and a discrete function. In other words, the mass of a wire can be separated into two parts; the part that is continuously distributed, and the “discrete” part that is contained in the beads.

The Set of Discontinuities of an Increasing Function
**Function**

Suppose that $\phi$ is an increasing function. Then the set of numbers at which $\phi$ is discontinuous is countable.

**Proof:** We write $E$ for the set of all numbers $x$ at which $\phi$ fails to be continuous. Given any number $x \in E$ we know that

$$\lim_{t \to x^-} f(t) < \lim_{t \to x^+} f(t)$$

and, using this fact, we choose a rational number that we shall call $r(x)$ such that

$$\lim_{t \to x^-} f(t) < r(x) < \lim_{t \to x^+} f(t).$$

In this way we have defined a one-one function $r$ from $E$ into the countable set $\mathbb{Q}$ of rational numbers.

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**Discrete Variation of an Increasing Function: Discrete Functions**

Suppose that $\phi$ is an increasing function, that $E$ is an elementary set and that $\{x_n \mid n \in \mathbb{Z}^+\}$ is the set of numbers in $E$ at which $\phi$ is discontinuous. The discrete variation of $\phi$ on $E$ is defined to be the number

$$\text{dvar}(\phi, E) = \sum_{n=1}^{\infty} J(\phi, x_n).$$

From the rearrangement theorem for series with nonnegative terms we know that the latter sum is independent of the order in which the discontinuities of $\phi$ appear.

An increasing function $\phi$ is said to vary discretely on an elementary set $E$ if

$$\text{dvar}(\phi, E) = \text{var}(\phi, E).$$

In other words, an increasing function $\phi$ varies discretely on an elementary set $E$ if $\text{var}(\phi, E)$ is the sum of the jumps that $\phi$ has in $E$. An increasing function $\phi$ that varies discretely on every elementary set is said to be a discrete function.

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**Some Properties of Discrete Variation**

Suppose that $\phi$ is an increasing function.

1. If $E$ is any elementary set then $\text{dvar}(\phi, E) \leq \text{var}(\phi, E)$.
2. If $E$ is any finite set then $\phi$ varies discretely on $E$.
3. If $A$ and $B$ are elementary sets that are disjoint from each other then

$$\text{dvar}(\phi, A \cup B) = \text{dvar}(\phi, A) + \text{dvar}(\phi, B).$$

4. If $\phi$ varies discretely on an elementary set $E$ then $\phi$ must vary discretely on every
elementary subset of $E$.

5. If $\phi$ varies discretely on each of two elementary sets $A$ and $B$ then $\phi$ must vary discretely on the set $A \cup B$.

6. The function $\phi$ is a discrete function if and only if for every open interval $(a,b)$ we have
   \[ \text{dvar}(\phi, (a,b)) = \phi(b-) - \phi(a+). \]

**Proof:** To prove part 1 we write the set of numbers in $E$ at which $\phi$ is discontinuous as
   \[ \{x_n \mid n \in \mathbb{Z}^+\} \]
and, to obtain a contradiction, we assume that
   \[ \sum_{n=1}^{\infty} J(\phi, x_n) > \text{var}(\phi, E). \]
Choose a positive integer $n$ such that
   \[ \sum_{j=1}^{n} J(\phi, x_j) > \text{var}(\phi, E). \]
Choose an interval $[a,b]$ that includes $E$ and choose a partition $P$ of the interval $[a,b]$ such that the function $\chi_E$ steps within $P$ and such that $P$ contains all of the numbers $x_1, x_2, x_3, \ldots, x_n$.

We now have
   \[ \text{var}(\phi, E) = \int_a^b \chi_E \, d\phi \geq \sum_{j=1}^{n} J(\phi, x_j) \]
which yields the required contradiction.

Now to prove part 2 we assume that $E$ is a finite set and we write
   \[ E = \{x_1, x_2, x_3, \ldots, x_n\}. \]
Then since
   \[ \text{var}(\phi, E) = \sum_{j=1}^{n} \text{var}(\phi, \{x_j\}) = \sum_{j=1}^{n} J(\phi, x_j) = \text{dvar}(\phi, E) \]
we know that $\phi$ varies discretely on $E$.

To prove part 3 we assume that $A$ and $B$ are elementary sets that are disjoint from each other and we write the set of numbers in $A \cup B$ at which $\phi$ is discontinuous as $\{x_n \mid n \in \mathbb{Z}^+\}$. We define
   \[ S = \{n \in \mathbb{Z}^+ \mid x_n \in A\} \quad \text{and} \quad T = \{n \in \mathbb{Z}^+ \mid x_n \in B\}. \]
Then by a property of subseries we know that
   \[ \text{dvar}(\phi, A \cup B) = \sum_{n=1}^{\infty} J(\phi, x_n) = \sum_{S} J(\phi, x_n) + \sum_{T} J(\phi, x_n) \]
   \[ = \text{dvar}(\phi, A) + \text{dvar}(\phi, B). \]
To prove part 4 we assume that \( \phi \) varies discretely on an elementary set \( A \) and that \( B \) is an elementary subset of \( A \). We define \( C = A \setminus B \). Then since
\[
\text{dvar}(\phi, A) = \text{var}(\phi, A) = \text{var}(\phi, B) + \text{var}(\phi, C) \\
\geq \text{dvar}(\phi, B) + \text{dvar}(\phi, C) = \text{dvar}(\phi, A)
\]
we deduce that \( \text{var}(\phi, B) = \text{dvar}(\phi, B) \) and also that \( \text{var}(\phi, C) = \text{dvar}(\phi, C) \).

To prove part 5 we suppose that \( A \) and \( B \) are elementary sets and that \( \phi \) varies discretely on both \( A \) and \( B \). From part 4 we deduce that \( \phi \) varies discretely on the set \( B \setminus A \). We see that
\[
\text{dvar}(\phi, A \cup B) = \text{dvar}(\phi, A) + \text{dvar}(\phi, B \setminus A) \\
= \text{var}(\phi, A) + \text{var}(\phi, B \setminus A) = \text{var}(\phi, A \cup B).
\]

Finally, to prove part 6 we note that the condition stated there is simply the condition that \( \phi \) should vary discretely on every open interval. Thus the condition certainly holds if \( \phi \) is a discrete function. Furthermore, if the condition holds then, since every elementary set is a subset of an open interval, we deduce from part 4 that \( \phi \) varies discretely on every elementary set.

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**Decomposing an Increasing Function into Discrete and Continuous Parts**

Suppose that \( \phi \) is an increasing function. There exist increasing functions \( \phi_1 \) and \( \phi_2 \) such that \( \phi_1 \) is discrete and \( \phi_2 \) is continuous and such that
\[
\phi = \phi_1 + \phi_2.
\]

**Proof:** Choose any real number \( c \). We define
\[
\phi_2(x) = \begin{cases} 
\text{var}(\phi,(c,x)) - \text{dvar}(\phi,(c,x)) & \text{if } x > c \\
0 & \text{if } x = c \\
-\text{var}(\phi,(c,x)) + \text{dvar}(\phi,(c,x)) & \text{if } x < c
\end{cases}
\]
and we now define \( \phi_1 = \phi - \phi_2 \). To complete the proof we need to show that the functions \( \phi_1 \) and \( \phi_2 \) are increasing, that \( \phi_1 \) is discrete and that the function \( \phi_2 \) is continuous. We begin with the observation that if \( t \) and \( x \) are any two numbers and \( t < x \) then
\[
\phi_2(x) - \phi_2(t) = \text{var}(\phi,(t,x)) - \text{dvar}(\phi,(t,x)).
\]

To understand why this identity holds we need to check each of the three cases

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\( c \leq t < x \)
For example, if $c \leq t < x$ then we have
\[
\phi_2(x) - \phi_2(t) = \varphi(\phi_s(t,c,x)) - \varphi(\phi_s(t,c)) - \varphi(\phi_s(t,x)) + \varphi(\phi_s(t,x))
\]
\[
= \varphi(\phi_s(t,x)) - \varphi(\phi_s(t,x))
\]
\[
= \varphi(\phi_s(t,x)) + J(\phi,t) - [\varphi(\phi_s(t,x)) + J(\phi,t)]
\]
\[
= \varphi(\phi_s(t,x)) - \varphi(\phi_s(t,x)).
\]
From the equation
\[
\phi_2(x) - \phi_2(t) = \varphi(\phi_s(t,x)) - \varphi(\phi_s(t,x))
\]
and this property of the variation functions we deduce that whenever $t < x$ we have
\[
\phi_2(x) - \phi_2(t) \geq 0
\]
which tells us that the function $\phi_2$ is increasing. The equation
\[
\phi_2(x) - \phi_2(t) = \varphi(\phi_s(t,x)) - \varphi(\phi_s(t,x))
\]
also tells us that when $t < x$ we have
\[
0 \leq \phi_2(x) - \phi_2(t) \leq \varphi(\phi_s(t,x))
\]
which we can write as
\[
0 \leq \phi_2(x) - \phi_2(t) \leq \varphi(x)-\varphi(t+).
\]
We shall now show that the function $\phi_2$ must be continuous. Suppose that $x$ is any real number. Given any number $t < x$ we have
\[
0 \leq \phi_2(x) - \phi_2(t) \leq \varphi(x)-\varphi(t+),
\]
\[
\phi_2(x) - \phi_2(t+),
\]
and since
\[
\lim_{t \to x^-}(\phi(x)-\phi(t)) = 0
\]
we deduce that
\[
\phi_2(x) - \phi_2(x-) = 0.
\]
By an almost identical argument we can see that
\[
\phi_2(x+) - \phi_2(x) = 0
\]
and so the function $\phi_2$ must be continuous at the number $x$. Therefore $\phi_2$ is a continuous function.

To see that the function $\phi_1$ is increasing we suppose that $t < x$. From the inequality
\[
0 \leq \phi_2(x) - \phi_2(t) \leq \varphi(x)-\varphi(t+)
\]
we see that
and we deduce that
\[ \phi_2(x) - \phi_2(t) \leq \phi(x) - \phi(t) \]

Now we look at the jumps of \( \phi_1 \). Given any number \( x \) we have
\[ \phi_1(x+) - \phi_1(x-) = \phi(x+) - \phi(x-) - [\phi_2(x+) - \phi_2(x-)] = \phi(x+) - \phi(x-). \]

Therefore
\[ J(\phi_1, x) = J(\phi, x) \]

for every number \( x \) and therefore if \( (a, b) \) is any open interval we have
\[ \text{dvar}(\phi_1, (a, b)) = \text{dvar}(\phi, (a, b)). \]

We can now use this property of variations to show that the function \( \sigma_1 \) is discrete. Suppose that \( (a, b) \) is an open interval. Then
\[
\text{var}(\phi_1, (a, b)) = \phi_1(b) - \phi_1(a +) \\
= \phi(b) - \phi(a +) - [\phi_2(b) - \phi_2(a)] \\
= \text{var}(\phi, (a, b)) - [\text{var}(\phi, (a, b)) - \text{dvar}(\phi, (a, b))] \\
= \text{dvar}(\phi, (a, b)) = \text{dvar}(\phi_1, (a, b)).
\]

Therefore \( \phi_1 \) is discrete, as promised. ■

### Uniqueness of the Decomposition

Suppose that an increasing function \( \phi \) is written in the form \( \phi_1 + \phi_2 \) and also as \( \psi_1 + \psi_2 \) where the functions \( \phi_1, \phi_2, \psi_1 \) and \( \psi_2 \) are increasing and the functions \( \phi_1 \) and \( \psi_1 \) are discrete and the functions \( \phi_2 \) and \( \psi_2 \) are continuous. Then for some constant \( c \) we have
\[ \phi_1 = \psi_1 + c \quad \text{and} \quad \phi_2 = \psi_2 - c. \]

**Proof:** Since
\[ \phi_1 - \psi_1 = \psi_2 - \phi_2, \]
the function \( \phi_1 - \psi_1 \) must be continuous. Therefore if \( x \) is any number we have
\[ J(\phi_1, x) - J(\psi_1, x) = (\phi_1 - \psi_1)(x+) - (\phi_1 - \psi_1)(x-) = 0. \]

Thus if \( a \) and \( b \) are any numbers and \( a < b \) we have
\[
(\phi_1 - \psi_1)(b) - (\phi_1 - \psi_1)(a) = (\phi_1 - \psi_1)(b) - (\phi_1 - \psi_1)(a +) \\
= \text{dvar}(\phi_1, (a, b)) - \text{dvar}(\psi_1, (a, b)) = 0
\]
and so the function \( \phi_1 - \psi_1 \) must be constant. Call the constant \( c \). ■

### Integration with Respect to a Discrete Function

We have already hinted that if \( \phi \) is an increasing function and \( f \) is a bounded function on
an interval $[a, b]$ then the integrability or lack of integrability of $f$ with respect to $\phi$ has nothing to do with the behavior of $f$ at the numbers at which $\phi$ is discontinuous. In this section we shall make this idea precise by observing that if $\phi$ is discrete then $f$ must always be integrable with respect to $\phi$. We shall see, in fact, that if $\{x_n \mid n \in \mathbb{Z}^+\}$ is the set of numbers in $[a, b]$ at which $\phi$ is discontinuous then

$$
\int_a^b f \, d\phi = \sum_{n=1}^{\infty} f(x_n) J(\phi, x_n).
$$

We begin by looking that the integral of a step function.

### Integration of a Step Function with Respect to a Discrete Function

Suppose that $\phi$ is an increasing function that varies discretely on an elementary set $E$ and that $f$ is a step function. Then if $\{x_n \mid n \in \mathbb{Z}^+\}$ is the set of numbers in $E$ at which $\phi$ is discontinuous then we have

$$
\int_E f \, d\phi = \sum_{n=1}^{\infty} f(x_n) J(\phi, x_n).
$$

**Proof:** We observe first that since

$$
\sum_{n=1}^{\infty} J(\phi, x_n) = \text{var}(\phi, E)
$$

the absolute convergence of the series $\sum f(x_n) J(\phi, x_n)$ follows from the comparison test. Suppose that the range of $f$ is the set $\{\alpha_1, \alpha_2, \ldots, \alpha_k\}$ and, for each $j$, suppose that

$$
E_j = \{x \in E \mid f(x) = \alpha_j\} \quad \text{and} \quad S_j = \{n \in \mathbb{Z}^+ \mid x_n \in E_j\}.
$$

We deduce from the additivity property of the integral and a theorem on rearrangements of series that

$$
\int_E f \, d\phi = \sum_{j=1}^{k} \int_{E_j} f \, d\phi = \sum_{j=1}^{k} \alpha_j \text{var}(\phi, E_j) = \sum_{j=1}^{k} \alpha_j \text{dvar}(\phi, E_j)
$$

$$
= \sum_{j=1}^{k} \alpha_j \sum_{n \in S_j} J(\phi, x_n) = \sum_{j=1}^{k} \sum_{n \in S_j} f(x_n) J(\phi, x_n) = \sum_{n=1}^{\infty} f(x_n) J(\phi, x_n).
$$

### Integrability with Respect to a Discrete Function
Suppose that an increasing function $\phi$ varies discretely on an interval $[a, b]$. Then every bounded function on $[a, b]$ is Riemann-Stieltjes integrable with respect to $\phi$.

**Proof:** We shall write the set of numbers in $[a, b]$ at which $\phi$ is discontinuous as $\{t_n \mid n \in \mathbb{Z}^+\}$. Since $\phi$ varies discretely on $[a, b]$ we have

$$\text{var}(\phi, [a, b]) = \sum_{n=1}^{\infty} J(\phi, t_n).$$

Now suppose that $f$ is any bounded function on the interval $[a, b]$. In order to show that $f$ is Riemann-Stieltjes integrable on $[a, b]$ we shall show that $f$ satisfies the **second criterion** for integrability. Suppose that $\varepsilon > 0$ and choose a positive integer $n$ such that

$$\sum_{j=n}^{\infty} J(\phi, t_j) < \varepsilon.$$

We now define $P$ to be the partition of $[a, b]$ whose points are the numbers $a$ and $b$ and the numbers $t_1, t_2, \ldots, t_n$ taken in ascending order, we rename this partition as $P = (x_0, x_1, \ldots, x_m)$ and we define

$$E = \bigcup_{j=1}^{m} (x_{j-1}, x_j).$$

We see that $w(P, f)(x) = 0$ whenever $x \in [a, b] \setminus E$ and we see that

$$\text{var}(\phi, E) = \text{var}(\phi, [a, b]) - \sum_{j=1}^{m} J(\phi, x_j) \leq \sum_{j=n}^{\infty} J(\phi, t_j) < \varepsilon.$$

Therefore $f$ satisfies the **second criterion**.

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**Integral with Respect to a Discrete Function**

Suppose that $\phi$ is an increasing function that varies discretely on an interval $[a, b]$ and that $\{x_n \mid n \in \mathbb{Z}^+\}$ is the set of numbers in $[a, b]$ at which $\phi$ is discontinuous. Then given a bounded function $f$ on $[a, b]$, we have

$$\int_a^b f d\phi = \sum_{n=1}^{\infty} f(x_n) J(\phi, x_n).$$

**Proof:** Given any step functions $s$ and $S$ on $[a, b]$ satisfying the inequality

$$s \leq f \leq S$$

we see from the theorem on integrability with respect to a discrete function that

$$\int_a^b s d\phi = \sum_{n=1}^{\infty} s(x_n) J(\phi, x_n) \leq \sum_{n=1}^{\infty} f(x_n) J(\phi, x_n) \leq \sum_{n=1}^{\infty} S(x_n) J(\phi, x_n) = \int_a^b S d\phi.$$
Integrability with Respect to an Increasing Function

Suppose that $f$ is a bounded function defined on an interval $[a, b]$ and that $\phi$ is an increasing function that has been expressed in the form:

$$\phi = \phi_1 + \phi_2$$

where $\phi_1$ and $\phi_2$ are increasing functions and $\phi_1$ is discrete and $\phi_2$ is continuous. Then $f$ is Riemann-Stieltjes integrable with respect to $\phi$ if and only if $f$ is Riemann-Stieltjes integrable with respect to $\phi_2$.

**Proof:** Since integrability of $f$ with respect to $\phi_1$ is automatic, the theorem follows at once from the linearity property of the integral. ■

The Junior Lebesgue Criterion Revisited

Suppose that $f$ is a bounded function defined on an interval $[a, b]$ and that $\phi$ is an increasing function that has been expressed in the form:

$$\phi = \phi_1 + \phi_2$$

where $\phi_1$ and $\phi_2$ are increasing functions and $\phi_1$ is discrete and $\phi_2$ is continuous. Then a sufficient condition for $f$ to be Riemann-Stieltjes integrable with respect to $\phi$ is that for every number $\varepsilon > 0$ there exists an elementary subset $E$ of $[a, b]$ such that $\text{var}(\phi_2, E) < \varepsilon$ and $f$ is continuous on the set $[a, b] \setminus E$.

The Lebesgue Criterion for Riemann-Stieltjes integrability

The full Lebesgue criterion for Riemann-Stieltjes integrability is discussed in the Riemann-Stieltjes version of the optional chapter on sets of measure zero.

Integration of Monotone Functions

Up till now we have been concerned with monotonicity as a key property of the function $\phi$ but in this section we shall show that monotonicity also has a role to play for the function $f$. We begin with the observation that all monotone functions are integrable.
Suppose that \( f \) is a monotone function defined on an interval \([a, b]\). Then \( f \) is Riemann-Stieltjes integrable with respect to every increasing function \( \phi \).

Proof: Suppose that \( \phi \) is an increasing function. We assume, without loss of generality that \( f \) is increasing and, with the above theorem on integrability in mind, we assume without loss of generality that the function \( \phi \) is continuous. Given any positive integer \( n \), we define \( P_n \) to be the regular \( n \)-partition of the interval \([\phi(a), \phi(b)]\). Thus if, for a given \( n \),

\[
P_n = (y_0, y_1, y_2, \ldots, y_n)
\]

then for each \( j \) we have

\[
y_j = \phi(a) + \frac{j(\phi(b) - \phi(a))}{n}.
\]

Using the fact that \( \phi \) is continuous and the Bolzano intermediate value theorem we choose a number \( x_j \in [a, b] \) for each \( j \) such that \( \phi(x_j) = y_j \). In this way we obtain a partition

\[
Q = (x_0, x_1, x_2, \ldots, x_n)
\]

of the interval \([a, b] \). We now define \( s_n \) and \( S_n \) to be the step functions on \([a, b] \) that take the value \( f(x_j) \) at each point \( x_j \) of \( Q \) and that take the constant values \( f(x_{j-1}) \) and \( f(x_j) \) respectively in each interval \((x_{j-1}, x_j) \). We observe that \( s_n \leq f \leq S_n \) and that

\[
\int_a^b (S_n - s_n) d\phi = \sum_{j=1}^n (f(x_j) - f(x_{j-1})) \frac{\phi(b) - \phi(a)}{n} = \frac{\phi(b) - \phi(a)}{n} \sum_{j=1}^n (f(x_j) - f(x_{j-1})) = \frac{(\phi(b) - \phi(a))(f(b) - f(a))}{n}
\]

and we conclude that

\[
\lim_{n \to \infty} \int_a^b (S_n - s_n) d\phi = 0.
\]

from which it follows that the pair of sequences \( (s_n) \) and \( (S_n) \) squeezes the function \( f \) with respect to \( \phi \) on \([a, b] \) and so \( f \) is integrable with respect to \( \phi \). \( \blacksquare \)

### Introduction to Integration by Parts

If \( u \) and \( v \) are increasing functions with Riemann integrable derivatives then the integration by parts formula

\[
\int_a^b u(x)v'(x)dx + \int_a^b v(x)u'(x)dx = u(b)v(b) - u(a)v(a)
\]
that we saw earlier can also be written in a form that involved Stieltjes integrals. We can use the relationship between Riemann integrals and Riemann-Stieltjes integrals to write
\[ du(x) = u'(x)dx \quad \text{and} \quad dv(x) = v'(x)dx \]
inside the integral sign and obtain the integration by parts formula in the form
\[ \int_a^b u(x)dv(x) + \int_a^b v(x)du(x) = u(b)v(b) - u(a)v(a) \]
which we can also write as
\[ \int_a^b udv + \int_a^b vdu = u(b)v(b) - u(a)v(a). \]
With the latter equation in mind we may ask whether if \( f \) and \( \phi \) are increasing functions on an interval \([a, b]\) then, considering that the Riemann-Stieltjes integrals \( \int_a^b f d\phi \) and \( \int_a^b \phi df \) both exist, the equation
\[ \int_a^b f d\phi + \int_a^b \phi df = f(b)\phi(b) - f(a)\phi(a) \]
is true whenever \( f \) and \( \phi \) are increasing functions on a given interval \([a, b]\). As we know, the two integrals on the left side will certainly exist.
Unfortunately, the answer to this general question is no. To see what can go wrong we look at the case in which
\[ f(x) = \phi(x) = \begin{cases} 0 & \text{if} \quad x < 1 \\ 1 & \text{if} \quad x \geq 1 \end{cases} \]
and we observe that, even though \( f \) and \( \phi \) are increasing functions on the interval \([0, 2]\), we have
\[ \int_0^2 f d\phi + \int_0^2 \phi df = 2 \neq f(2)\phi(2) - f(0)\phi(0). \]
The pathology exhibited by this example disappears, however, if we require that at every number \( x \in [a, b] \) we have
\[ f(x)J(\phi, x) + \phi(x)J(f, x) = f(x+)\phi(x+) - f(x-)\phi(x-). \]
If this condition holds then the integration by parts formula takes on the form
\[ \int_a^b f d\phi + \int_a^b \phi df = f(b+)\phi(b+) - f(a-)\phi(a-). \]
At first sight the equation
\[ f(x)J(\phi, x) + \phi(x)J(f, x) = f(x+)\phi(x+) - f(x-)\phi(x-) \]
may look a little artificial but it is really quite natural, as the following three observations show:

1. The equation
\[ f(x)J(\phi, x) + \phi(x)J(f, x) = f(x+)\phi(x+) - f(x-)\phi(x-) \]
can be written in the form
\[ \int_x^x f \, d\phi + \int_x^x \phi \, df = f(x+)\phi(x+) - f(x-)\phi(x-) \]

and it can therefore looked upon as being a “mini-parts” formula on the interval \([x,x]\). We are therefore saying that the parts formula holds on the interval \([a,b]\) as long as it holds on each singleton in \([a,b]\).

2. The condition

\[ f(x)J(\phi,x) + \phi(x)J(f,x) = f(x+)\phi(x+) - f(x-)\phi(x-) \]

holds automatically at any number \(x\) at which either of the functions \(f\) and \(\phi\) is continuous. In fact, we can say a little more: The condition

\[ f(x)J(\phi,x) + \phi(x)J(f,x) = f(x+)\phi(x+) - f(x-)\phi(x-) \]

holds at any number \(x\) at which at least one of the two functions is continuous from the left and at least one of the two functions is continuous from the right.

3. The condition

\[ f(x)J(\phi,x) + \phi(x)J(f,x) = f(x+)\phi(x+) - f(x-)\phi(x-) \]

holds at any number \(x\) at which the function value \(f(x)\) is the arithmetic mean of the numbers \(f(x-)\) and \(f(x+)\) and the function value \(\phi(x)\) is the arithmetic mean of the numbers \(\phi(x-)\) and \(\phi(x+)\).

The technical details that are needed for a proof of the integration by parts identity

\[ \int_a^b f \, d\phi + \int_a^b \phi \, df = f(b+)\phi(b+) - f(a-)\phi(a-) \]

are contained in the following lemma.

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**An Integration by Parts Identity Involving Upper and Lower Functions**

Suppose that \(f\) and \(\phi\) are increasing functions on an interval \([a,b]\) and that the condition

\[ f(x)J(\phi,x) + \phi(x)J(f,x) = f(x+)\phi(x+) - f(x-)\phi(x-) \]

holds at every number \(x\) in the interval \([a,b]\). Then given any partition \(P\) of \([a,b]\), we have

\[ \int_a^b l(P,f) \, d\phi + \int_a^b u(P,\phi) \, df = f(b+)\phi(b+) - f(a-)\phi(a-) \]

**Proof:** We note first that for each \(j\), the constant values of the functions \(l(P,f)\) and \(u(P,\phi)\) in the interval \((x_{j-1},x_j)\) are \(f(x_{j-1}+)\) and \(\phi(x_j-)\) respectively. Therefore the expression

\[ \int_a^b l(P,f) \, d\phi + \int_a^b u(P,\phi) \, df \]

is equal to
\[
\sum_{j=0}^{n} f(x_j)J(\phi, x_j) + \sum_{j=1}^{n} f(x_{j-1} +)[\phi(x_j) - \phi(x_{j-1} +)] + \sum_{j=0}^{n} \phi(x_j)J(f, x_j) + \sum_{j=1}^{n} \phi(x_j)\left[f(x_j) - f(x_{j-1} +)\right]
\]

\[
= \sum_{j=0}^{n} [f(x_j)J(\phi, x_j) + \phi(x_j)J(f, x_j)] + \sum_{j=1}^{n} [f(x_j) - f(x_{j-1} +)\phi(x_{j-1} +)]
\]

\[
= \sum_{j=0}^{n} [f(x_j +)\phi(x_j +) - f(x_j -)\phi(x_j -)] + \sum_{j=1}^{n} [f(x_j -)\phi(x_{j-1} +) - f(x_{j-1} +)\phi(x_{j-1} +)]
\]

\[
= f(b +)\phi(b +) - f(a -)\phi(a -).
\]

The General Integration by Parts Identity

Suppose that \(f\) and \(\phi\) are increasing functions on an interval \([a, b]\) and that the condition

\[
f(x)J(\phi, x) + \phi(x)J(f, x) = f(x +)\phi(x +) - f(x -)\phi(x -)
\]

holds at every number \(x\) in \([a, b]\). Then

\[
\int_{a}^{b} f \, d\phi + \int_{a}^{b} \phi \, df = f(b +)\phi(b +) - f(a -)\phi(a -).
\]

**Proof:** We already know that the two integrals on the left side must exist. Given any number \(\varepsilon > 0\), we can use the fact that these integrals exist and the second criterion for integrability to choose partitions \(P_1\) and \(P_2\) of \([a, b]\) such that

\[
\int_{a}^{b} w(P_1, f) \, d\phi < \varepsilon \quad \text{and} \quad \int_{a}^{b} w(P_2, \phi) \, df < \varepsilon
\]

and, by defining \(P\) to be the common refinement of \(P_1\) and \(P_2\), we obtain

\[
\int_{a}^{b} w(P, f) \, d\phi < \varepsilon \quad \text{and} \quad \int_{a}^{b} w(P, \phi) \, df < \varepsilon.
\]

We can therefore choose a sequence \((P_n)\) of partitions of \([a, b]\) such that the inequalities

\[
\int_{a}^{b} w(P_n, f) \, d\phi < \frac{1}{n} \quad \text{and} \quad \int_{a}^{b} w(P_n, \phi) \, df < \frac{1}{n}
\]

hold for every positive integer \(n\). Thus

\[
\int_{a}^{b} f \, d\phi + \int_{a}^{b} \phi \, df = \lim_{n \to \infty} \left( \int_{a}^{b} l(P_n, f) \, d\phi + \int_{a}^{b} u(P_n, \phi) \, df \right) = f(b +)\phi(b +) - f(a -)\phi(a -).
\]

Exercises that Yield Another Version of Darboux's Theorem
Homework

The exercises in this section depend upon the material on Darboux’s theorem. Their purpose is to show that if \( f \) is Riemann-Stieltjes integrable with respect to an increasing function \( \phi \) and if \( f \) is continuous at every number at which \( \phi \) jumps then Darboux’s theorem can be stated using the ordinary mesh \( \|P\| \) of a partition \( P \) instead of the \( \phi \)-mesh.

1. Prove the following special case of Darboux’s theorem that applies when \( \phi \) is continuous: Suppose that \( f \) is Riemann-Stieltjes integrable with respect to an increasing continuous function \( \phi \) on an interval \([a, b]\). Then for every number \( \varepsilon > 0 \) there exists a number \( \delta > 0 \) such that for every partition

\[
P = (x_0, x_1, \ldots, x_n)
\]

of \([a, b]\) satisfying the inequality \( \|P\| < \delta \) and every choice of numbers \( t_j \in [x_{j-1}, x_j] \) for each \( j \) we have

\[
\left| \sum_{j=1}^{n} f(t_j)(\phi(x_j) - \phi(x_{j-1})) - \int_{a}^{b} f \, d\phi \right| < \varepsilon.
\]

2. Suppose that \( \phi \) is an increasing function that varies discretely on an interval \([a, b]\).

Suppose that \( f \) is a bounded function on \([a, b]\) and that \( f \) is continuous at every number at which the function \( \phi \) is discontinuous. Prove that for every number \( \varepsilon > 0 \) there exists a number \( \delta > 0 \) such that for every partition

\[
P = (x_0, x_1, \ldots, x_n)
\]

of \([a, b]\) satisfying the inequality \( \|P\| < \delta \) and for every choice of numbers \( t_j \in [x_{j-1}, x_j] \) for each \( j \), if we define \( g \) to be the step function that takes the value \( f(x_j) \) in each interval \((x_{j-1}, x_j)\) then we have

\[
\left| \int_{a}^{b} g \, d\phi - \int_{a}^{b} f \, d\phi \right| < \varepsilon.
\]

3. By combining the preceding two exercises, obtain an analog of Exercise 2 that does not require the assumption that \( \phi \) varies discretely on \([a, b]\).