Telling the Truth About Teaching Effectiveness

by

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Prologue

Everyone talks about “teaching effectiveness” but hardly anyone says anything. I think it’s time that someone told the truth. Instead of using this undefined concept as a yardstick for personnel action, we should start trying to achieve it.

I do not expect everyone to agree with the views I have set forth in this document and, in fact, I think that it is perfectly acceptable for us to disagree. What I think is not acceptable is that we should neither agree nor disagree because we have never communicated with one another about these important issues.

The Purpose of This Document

For several centuries, faculty at educational institutions have been deeply concerned with the role that these institutions play to ensure that the ample page of knowledge, rich with the spoils of time, will unroll to the eyes of the next generation and keep liquid the genial current of its soul. Teaching effectiveness should be our number one concern. Indeed, many have argued that one of the principal reasons for a requirement of scholarly pursuit among university faculty is merely to ensure that these faculty will display the enthusiasm and possess the insight and depth that are a necessary ingredient in an effective teacher.

In an institution such as Kennesaw State University where academic research is not the primary goal but is, instead, just one of several important qualities that we seek in our faculty, the issue of teaching effectiveness is paramount. It occupies first place in our thoughts throughout our careers and not a day goes by in which we are not working on ways to serve our students and our community more effectively. We, as academics, welcome every opportunity to communicate with others on this important issue, to seek an ever clearer definition of what teaching effectiveness means to us, and to seek ways in which we can work together to achieve ever higher levels of teaching effectiveness.

My reason for writing this document is my concern that, over the past few decades, the important issue of teaching effectiveness has been gradually hijacked by people who are
not, themselves, an active part of the academic teaching community and for whom teaching effectiveness is not as important as the role that it can play in personnel action. Instead of being free to apply their entire efforts to advance the knowledge, understanding, and future success of their students, faculty have been subject, in recent years, to an ever increasing atmosphere of personnel assessment. The highly complicated subject of teaching effectiveness has been allowed to degenerate into the production of shallow simplistic data that is more a measure of consumer satisfaction and customer appreciation than it is a true measure of the quality of instruction in our classrooms.

I believe that we, as academics, have the responsibility to take a stand against any activity that threatens to erode the quality of our instruction and the ability of the next generation to succeed. It is our responsibility to make a clear distinction between activity that is designed to advance teaching effectiveness and activity that is designed merely to give an appearance of commitment to teaching effectiveness, between activity designed to advance teaching effectiveness and activity designed to use the issue of teaching effectiveness as a benchmark for promotion and tenure considerations. Above all, it is our responsibility to ensure that an academic will never feel pressured to take an unprofessional course of action, one that would reduce his or her teaching effectiveness, just for the sake of being seen by others as a more effective instructor.

To those people outside the academic world and to those people in the administrations of our educational institutions who want to talk to us about the assessment of teaching effectiveness we need to answer: “Show us first that your primary concern is teaching effectiveness and not merely the measurement of teaching effectiveness. Show us first that your primary objective is to find a strategy that will actually improve teaching effectiveness rather than merely to find a litmus test for evaluating faculty portfolios.” In other words, the measurement of teaching effectiveness will become meaningful when our commitment to teaching effectiveness is beyond doubt and we have begun to work together on our primary objective which is the actual achievement of teaching effectiveness.

In this document I shall give my perspective on these important matters and I shall look critically at some of the traditional ways in which institutions have attempted to measure teaching effectiveness in recent years.

The General Principles

The Importance of a Common Purpose

I believe that, before we can work towards a meaningful definition of teaching effectiveness and, before we can effect a successful strategy for enhancing it, we need to ensure that students and instructors will walk into the classroom with the same purpose in mind. For such a common purpose to exist, our students must feel confident that every course they take with us will be relevant to them and will present an integral part of what they need to know for their chosen fields of study. We have to ensure that the student and the instructor will never be at cross purposes in the classroom. They must be striving to achieve the same goals.
To provide this feeling of confidence, it may be necessary for us to make some far reaching, painful, and fundamental changes to university curricula so that we can demonstrate our solid commitment to the principle that every course a student is compelled to take will be relevant to and needed by the student’s major field of study. We need to be conscious of the enormous effort, both human and financial, that a student must make for every course that he or she takes and we need to be able to promise that the benefit of taking the course is worth that effort.

I certainly believe that universities should offer a broad spectrum of courses and that they should encourage students to consider taking courses that will broaden their horizons. However, I do not believe that we should be compelling students to take such courses and I do not believe that the grades earned in such courses should contribute to a student GPA. Those of us who are truly committed to teaching excellence need to come to terms with the reality that the core curriculum is a multibillion dollar industry that sits like a millstone around the neck of academic life in the United States of America. It is a gross waste, both of taxpayers’ money and student money and it serves to inflate both the size and number of the nation’s educational institutions. But for the core, we could reduce many of our degree programs to three years instead of four. But for the core, we could focus both our students’ efforts and our own efforts on activity that is actually relevant to the students’ future success. If some of our institutions are rendered redundant by abolition of the core, we should convert them into vocational schools that would make much better use of taxpayers’ money and enable people to make a better life for themselves.

Of course we expect universities to broaden the horizons of their students. To achieve this end, we should make cultural activity available and encourage people to partake of it, if they so desire, but we do not achieve anything by ramming culture down people’s throats. With all due respect, we do not cause people to appreciate music by compelling them to take a course for grade on appreciation of music. When I was an undergraduate at the University of the Witwatersrand where there was no core curriculum, we experienced a rich cultural campus life. And who did one find at, say, a flute concert given by Jean Pierre Rampal? The answer is, students of mathematics, physics, medicine, dentistry, and mechanical engineering.

It is well known that some of the finest doctors in the USA are expatriate South Africans who qualified at the University of the Witwatersrand Medical School. As freshmen, they registered as students of medicine and, after a single year of science courses (physics, chemistry, and biology), they moved “up the hill” to the medical school. Their second year covered anatomy and physiology. At the end of their sixth year, they were doctors, and very good doctors too. There was no need for a “pre-med” degree that would have served only to waste their time.

In the absence of a proper commitment to relevance in our university courses, we who teach mathematics face a very real problem. The structure of the university curriculum has forced some of our students to enter our courses with the primary objective of earning a course credit rather than with the primary objective of learning mathematics. We have no guarantee that every student who enters a mathematics course actually needs or wants to know the material. In fact, many of our students enter our courses with a slight feeling of resentment and, when such students exhibit dissatisfaction, we have to be careful not to
blame their instructors.

In saying this, I think of the case of a student I once had in Calculus I at the University of Wisconsin, Oshkosh. This young man was in desperate straits with little prospect of passing my calculus course and, indeed, had little desire to do so. I asked him what his major was and he surprised me by answering that it was *funeral directing*. “Why” I asked him, “are you taking calculus?” He answered that Calculus I was a requirement imposed on him by the university and by the Chicago Society of Funeral Directors. We can only imagine what sort of answer such a student would provide when filling a questionnaire that asks whether he is satisfied with the calculus course.

I think also about a student I had some years ago at Kennesaw State University. This young man had recently played football for Georgia Tech and was a highly competent athlete. He was also one of my very best mathematics students and earned a resounding A in my invitation-only second level real analysis course. However, this student was forced to delay the date of his graduation because it was discovered that he had not completed all of his P.E. courses. He was compelled to enter a course in Spanish Dancing before they would let him graduate.

I believe that, when the day comes on which we have achieved a common purpose with our students in all of their courses, those nasty little tables that spring up at the end of each term inviting students to sell back their textbooks will be just an unpleasant memory.

**The Importance of Proper Time Management**

If we are to achieve true teaching effectiveness in our courses, we have to know that our students have the time and the energy that they require to go beyond mere preparation for tests and examinations and to immerse themselves in a meaningful and enthusiastic study of the course material.

Unfortunately, all too often, they have registered for too many course credits and this problem is aggravated by the fact that most of our students combine full time study with the work they must do to remain financially solvent. It is only to be expected that many students in this category will set their sights on instructors who present their courses in a mechanical way. They will seek out and reward instructors who allow them to memorize procedures for producing right answers rather than encouraging them to understand the principles, to understand how to motivate the methods that are being used, and to understand why the methods are valid.

If we are to achieve true teaching effectiveness, then we have to come to terms with the reality that being a full time student is already a full time activity. Students who have to earn a living should be encouraged to study part time and we should be working constantly to make it easier for students to study in this way. Instead, what has the university done? According to information that has come my way, the university has recently *raised* the number of credits for which a student must be registered in order to qualify for student health insurance. With all due respect I have to say that this action is nothing short of a disgrace and I think that those who were involved with that decision should not dare, for shame, to talk to us about teaching effectiveness.
The Role of Student Evaluations of Instructors

The use of student evaluations to measure the quality of a course and the quality of an instructor is one of the most troublesome issues that confront us in any discussion of teaching effectiveness. Administrators and others who are not part of the actual teaching community have come to accept the value of student evaluations of their instructors in a totally uncritical way and instructors have been largely intimidated into silence on this issue. There is an automatic assumption that any instructor who dares to criticize the way in which student evaluations are conducted and used must have something to hide. We are not supposed to object to the indignity that is inflicted on us and on our profession when student evaluations have to be submitted anonymously and out of sight of the instructor. I myself found it necessary to hold my tongue on this subject for many years and it is only now that I am an old and grey tenured full professor with a long record of teaching successes and undoubted popularity with students that I dare raise this subject. For the most part, to speak against the gathering of anonymous student evaluations is just like saying that the emperor has no clothes.

I must emphasize that I am not against the principle of asking students for their opinions and asking them to express their state of mind. I believe that, properly administered, the gathering of student opinions can be a valuable resource to us in our quest of ever higher standards of teaching effectiveness. However, when the gathering of student opinions is performed more for personnel action purposes than to enhance teaching effectiveness then it becomes problematic. Furthermore, when students are given to understand that, in submitting evaluations, they are being given the power of life or death on the careers of their instructors, then I believe that the process has become very unhealthy and damaging and an affront to our professional dignity.

My own method of choice for gathering student opinions is to hold periodic round table discussions with my students. Students see such discussions as what they are, an attempt to gather information that will help us to serve them better. There is nothing hostile about our relationship during such discussions and, when we engage in dialogue with one another, the students learn as much about me as I learn about them. Over the years, these discussions have enabled me to make important improvements to my pedagogical
technique but they have also served to clarify misunderstandings that students have. By contrast, the anonymous evaluations that we are all required to conduct presuppose a hostile relationship. Because these evaluations are submitted anonymously and are not subject to cross questioning and because the instructor is not even allowed to be present, they are, as I have said, an affront to the dignity of our profession. They also send the wrong message to our students. Students should not be led to believe that they are experts on the subject matter of their courses or that they are experts on matters of pedagogy. We instructors, who are highly talented individuals who have devoted years of our lives to the production of quality instruction deserve a little trust and respect. We do not deserve to have this respect undermined by the evaluation process and we do not deserve the indignity that is inflicted on us when we have to leave to leave the classroom and stand out in the corridor while students are filling in the evaluation sheets.

Even if we were to accept the premise that students have the expertise and objectivity necessary to provide useful information in the anonymous evaluation process, the fact that the evaluations are anonymous renders them almost useless for a discipline like mathematics because we are unable to track them and to see the connection between student opinion and their actual success or failure in subsequent courses. Because what they say in the forms is not subject to cross questioning, it is totally unreliable and so all the indignity that we have to suffer is for nothing. What the students convey in the evaluation sheets is merely their momentary state of happiness, not what they have learned, and it certainly does not tell us why they have or have not learned the material of the course. As a result, instructors may be pressured to make policy decisions that are designed to optimize their ratings in the evaluation sheets rather than because they are professionally sound. If we are to permit these anonymous evaluations to continue, then we need to ask ourselves whether the primary task of an instructor is to make students feel happy or to make students successful.

Above all, we need to establish that commitment to teaching effectiveness and to the enhancement of teaching effectiveness is our principal objective and is a prerequisite to any process of assessing the effectiveness of our personnel. Instead of trying to come up with the “right questions” to ask students to answer in the evaluation forms, we need to take a long critical look at the process as a whole. In short, we need to stop looking at the process blindly and start looking at it intelligently.

Teaching Effectiveness in Mathematics

In this section of the document, I shall be commenting on some basic principles that, I believe, are fundamental ingredients in an effective mathematics instructor.

Assessment of Teaching Effectiveness in Mathematics

More than any other discipline, mathematics depends on the understanding that students achieved in earlier courses and the principal task of any course is to prepare students for success in the levels that will follow it. Thus mathematics courses are never isolated islands of activity. Every course builds on those that precede it and paves the way to more advanced study or to applications in other disciplines. Since the quality of instruction that
can be provided in any course depends on the students’ state of readiness for the course, a knowledge of this state of readiness is needed before we can evaluate that quality of instruction. Students who enter a course unprepared are not in a position to assess the extent to which their lack of proper preparation is responsible for their present misery.

Since the principal duty of an instructor in a mathematics course is to ensure the success of students in the courses that will follow, our most important consideration in any method of assessing teaching effectiveness must be the ability of an instructor to produce students who have future success. As important as it may be to engage in round table discussions with students about a course that they are presently taking, it is much more important to communicate with those students six months or a year later when their perspectives have broadened.

We must also know to what extent an instructor’s presentation is forwards compatible with courses that the students will take in the future. An important requirement for such forwards compatibility is that instructors in elementary courses need to be familiar with the material of the advanced courses. And, most important of all, instructors in any mathematics course need to ensure that students are learning the principles and not just learning how to come up with correct “answers”. To ensure that students are engaging in a properly balanced study of mathematics, we need to ensure that both an instructors presentation and his/her examination technique encourage that balanced study. We must work urgently to expunge “answer in a box” and “multiple choice” testing.

We in the mathematics community face the very real danger that we may reward instructors who succumb to the temptation to present their courses mechanically, who present mathematics merely as an anthology of memorized procedures and formulas, who pay little or no attention to the production of meaningful mathematical language, who are committed slavishly to the over use of hand-held calculators and computer algebra systems, and who paint their students into a corner that is revealed to them only two or three semesters down the line.

We need to ensure that our mathematics courses conform to fundamental principles of sound mathematics teaching such as the following:

1. *Being right is nothing in mathematics. Knowing why you are right is everything.*

2. *We do not solve a problem to find the answer. We seek the answer so that we can solve the problem.*

3. *It is not the purpose of a lecture to tell you how to do the homework. It is the purpose of homework to help you to study the lecture.*

4. *A person who cannot find the words to give a beautiful and clear explanation of a mathematical idea is not having a problem with words. That person does not understand the mathematics.*

One might argue that I have overstated my case. I do accept that a mathematics course needs to adhere to a delicate balance between the learning of methods that will produce “answers” and the learning of principles that will motivate and justify those methods. If I did overstate my case then I did so deliberately because of the tradition in our community of placing too strong a focus on the memorization of methods at the expense of the actual learning of mathematics.
A Little Hi Fi Story

In 1966, I walked into an electronics store in downtown Johannesburg with the purpose of buying a turntable for the playing of vinyl records. The turntable was powered by an electric motor that made it turn at $33 \frac{1}{3}$ revolutions per minute. My question to the salesman was whether I could take that turntable with me to the United States which has 110 V and 60 Hz electric current instead of the 220 V and 50 Hz current that exists in South Africa.

The salesman gave me the happy news that the turntable could be adapted for use in the USA. I was instructed to flip the voltage switch at the back of the machine from 220 V to 110 V and I was to change the position of a part inside the machine to adjust for the fact that, in the USA, the electric motor which be turning at $\frac{6}{5}$ of the speed at which it turned in South Africa. All of this made sense.

Then he said something that surprised me. On the top of the turntable there was a stroboscope, a little aluminium disk with lines on it. When the turntable was turning at exactly $33 \frac{1}{3}$ revolutions per minute, the lines appeared to be still. This made it possible to fine-tune the rate at which the turntable turns. The salesman instructed me to flip this disk over when I arrived in the USA so that it would show the slightly different pattern that would be needed for 60 Hz current.

I objected: “What does this little disk care about what kind of electricity is being used to drive the motor?” I asked. I simply could not understand why the disk would need to be flipped over when I arrived in the United States. So he explained the issue to me again very slowly: “When you use the turntable in South Africa, you put it this way. When you use it in the USA, you put it that way.”

“Well, yes, I hear you.” I said. “But I don’t understand why. Look here, I could have a camel walking around driving this turntable instead of an electric motor. Why should that matter? Either the turntable is or it is not turning at the necessary $33 \frac{1}{3}$ revolutions per minute. Why does this disk need to be flipped over??”

Again, the long suffering salesman explained the problem to me, even more slowly since I was clearly some kind of a dunce. “When you use the turntable in South Africa, you put it this way. When you use it in the USA, you put it that way.”

When I told him yet again that I didn’t understand, he realized that he was dealing with a problem customer and called the manager. I put my question to the manager and he smiled. He didn’t say a word. All he did was point up at the electric light above our heads. “Thank you so much!!” I said. “I understand now.” The salesman stood there totally bewildered. He clearly had no idea what was going on and I have no doubt that, to this day, he has no idea why it was necessary to flip that disk over when I arrived in the United States.

I mention this story because this is exactly how many instructors attempt to teach mathematics and we are in danger of rewarding them for teaching in this way.

The Balance Between Acquired Skills and Understanding

As I have said, the content in a sequence of mathematics courses requires a delicate
balance between the learning of skills that will produce answers and the attainment of understanding of mathematical principles. Actually, these two objectives go hand in hand. A person who goes out into the world of science, technology, or economics and who then finds it necessary to make use of mathematical techniques will be at a decided disadvantage if his/her mathematics education was confined largely to the memorization of mechanical procedures. No problem out there in the real world will be an exact match to a problem previously memorized from an undergraduate textbook and those members of the team who lack the insight that depends on an understanding of the mathematical principles will not be contributing members of the team. They will probably make the coffee while others do the work, and they will be paid accordingly.

So, even for those who have no announced interest in continued study in mathematics, a delicate balance must be preserved. I believe that the teaching effectiveness of our community is being severely undermined by an inadequate approach to the mathematical principles and that we encourage this lack of balance and reward it. Oddly enough, our courses also contain little spurts of lip service to the principles in which the pendulum has swung too far the other way. These little spurts are short lived and go nowhere and the students soon see that these little spurts are not part of the big picture. I shall give some examples of these below and it will become clear that, although most of my criticism of mathematics teaching is focussed on its over emphasis of mechanical procedures, I am also criticizing insincere and inconsistent swings of the pendulum on the other direction.

**Inconsistency in Our Approach to Functions**

It is a common and necessary practice in mathematics to step away from the strict definition of a mathematical concept when our work with that concept is being applied to another topic. The following examples illustrate what I mean:

1. The strict definition of a natural number at advanced level is that a natural number is an ordinal that is not a limit ordinal and is not preceded by any limit ordinal. Well, of course, no one would dream of looking at natural numbers this way in an elementary course. Just imagine the confusion if we said that $0 = \emptyset$ or if we said that $3 \in 7$ or if we said that $5 = 4 \cup \{4\} = \{0,1,2,3,4\}$.

2. The strict definition of an integer at advanced level is that it is an equivalence class of ordered pairs of natural numbers modulo the equivalence relation $\sim$ defined as $(a,b) \sim (c,d)$ if and only if $a + d = b + c$. Of course, no one would dream of looking at integers this way in an elementary course just as no one would dream of looking at a rational number as an equivalence class of ordered pairs of integers and no one would dream of looking at a real number as a Dedekind cut in the system of rational numbers.

So why on earth do we introduce the concept of a function by saying that a function is a set of ordered pairs???

Of course, the strict definition of a function is that it is a set $f$ every member of which is an ordered pair and that satisfies the condition that, whenever two ordered pairs $(x,y_1)$ and $(x,y_2)$ both belong to $f$, we must have $y_1 = y_2$. I think that it is absolutely absurd to introduce functions in this kind of way in elementary mathematics. To do so muddies the water and erodes our students’ confidence in our integrity because they soon discover that, as we progress, that we do not think of a function in this kind of way. The very use of the
word *graph* exposes the lie that we are telling because the graph of a function \( f \) is defined to be the set of ordered pairs \((x, y)\) for which \( y = f(x) \). In other words, the graph of a function is exactly what we were originally giving as the definition of the function itself.

Oddly enough, the very people and the very textbooks that begin the treatment of functions in this way soon abandon it. Quite soon they are telling us that \( x^2 \) is a function. If we ask them what \( x \) is, they will probably tell us that \( x \) is any number. Well, \( 3 \) is a number, so that means that \( 3^2 \) is a function, right?

I am not nitpicking here. If we speak in nonsense language, then we impose two tasks upon our students. First they have to learn how to translate what we say to translate what we have said into what we actually mean and then they have to get on with the mathematics itself. The very worst message we can give our students is that we are unreliable and that we do not always mean what we say.

I notice also that many publications that masquerade as elementary textbooks contain exercises that ask the student to find the domain of a function. Such exercises are meaningless because, until the domain of a function has been specified, the function has not been given a definition.

So, for example, if we were told that

\[
f(x) = x^2
\]

for every real number \( x \) then we could answer that the domain of the function is the set of all real numbers.

And, if we were told that

\[
f(x) = \frac{1}{x - 1}
\]

for every real number \( x \geq 5 \) then we could answer that the domain of the function is the set of all real numbers \( x \) for which \( x \geq 5 \).

However, if we are merely told that \( f(x) = x^2 \) then I would have to ask: *What is \( x \)?* Again, I am not nitpicking. As we talk nonsense in elementary mathematics courses, we train our students to think wrongly and we make it necessary for them to unlearn their bad habits when they come into more advanced courses.

**Solving Quadratic Equations**

The approach to the solution of quadratic equations in an elementary algebra course provides a good illustration of my comments about approaching mathematics mechanically without regard to principles and motivation. Instead of teaching students how to solve quadratic equations of the type

\[
ax^2 + bx + c = 0
\]

some instructors sidestep the topic by quoting the nearly worthless quadratic formula

\[
x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}.
\]

At its worst, this mechanical method may degenerate into a “Musgrave” ritual of typing the coefficients \( a, b, \) and \( c \) of a given quadratic into a hand-held calculator and then pressing a button to produce the “answer”. This mechanical method may welcomed by some students and it may optimize the instructor’ evaluation ratings but it leaves the students without any
understanding of the topic. The damage done by such “popular” teachers will not become apparent until the next level of study where the instructor will be downgraded by the students for not being as “good” as the previous instructor.

For example, a student who was trained only to quote the quadratic formula in an elementary algebra course, instead of learning how to complete the square will have a miserable time when evaluating the integral

$$\int_{1/2}^{2} \frac{1}{(2x^2 - 2x + 5)^{3/2}} \, dx$$

in Calculus II.

*Heron’s Formula*

Heron’s formula (sometimes known as Hero’s formula) is the method for calculating the area of a triangle with three known sides. The method involves use of the law of cosines to find the cosine of one of the angles in the triangle, and then uses the identity $\cos^2 \theta + \sin^2 \theta = 1$ to find the sine of that angle. At this point, the area of the triangle can be expressed as one half the product of the sine of this angle and the two sides of the triangle that flank it.

![Diagram of a triangle](image)

$$\text{area}(\triangle ABC) = \frac{1}{2} ab \sin \angle C$$

A few years ago, during the course of a Scientific Notebook workshop that I was presenting for a high school district in Massachusetts, I demonstrated the use Heron’s formula as an exercise in the production of a mathematical document. My exercise was to find the area of a triangle with sides 2, 3, and 4.

I found that, to everyone else in the room, Heron’s formula consisted of an actual rote memorized formula

$$\text{area}(\triangle ABC) = \sqrt{p(p-a)(p-b)(p-c)}$$

where $p$ is defined to be one half the sum $a + b + c$. Every one of the mathematics teachers in my workshop saw the problem as calculating the number

$$\sqrt{\left( \frac{2 + 3 + 4}{2} \right) \left( \frac{2 + 3 + 4}{2} - 2 \right) \left( \frac{2 + 3 + 4}{2} - 3 \right) \left( \frac{2 + 3 + 4}{2} - 4 \right)} = \frac{3\sqrt{15}}{4}.$$ 

They were a bit shocked at my admission that I couldn’t have quoted that formula if my life depended on it and I found to my amazement that, although they could all quote the formula, not one of these mathematics teachers had the slightest idea how to find the area
of the given triangle. It was a revelation to them to see me actually find the area instead of just reading it from a memorized formula.

In the workshop we used the law of cosines to show that

\[ 4^2 = 3^2 + 2^2 - (2)(2)(3) \cos \angle C \]

which gave us

\[ \cos \angle C = -\frac{1}{4} \]

and the equation

\[ \cos^2 \angle C + \sin^2 \angle C = 1 \]

then told us that

\[ \sin \angle C = \frac{\sqrt{15}}{4} \]

Then we concluded that

\[ \text{area}(\triangle ABC) = \frac{1}{2} (2)(3) \sin \angle C = \frac{3 \sqrt{15}}{4} \].

With an effort, I resisted the urge to ask them if they could derive the law of cosines.

**The Laws of Logarithms**

In too many elementary mathematics courses the laws of logarithms are quoted without proof and then the students are rushed headlong into homework problems. I am referring to the laws that say that

\[ \log_a (xy) = \log_a x + \log_a y \]
\[ \log_a \left(\frac{x}{y}\right) = \log_a x - \log_a y \]
\[ \log_a (x^p) = p \log_a x \]
\[ \log_a x = \frac{\log_b x}{\log_b a} \]

whenever \(a\) and \(b\) and \(x\) and \(y\) are positive and \(a\) and \(b\) are unequal to \(1\).

Some instructors will protest what I have just said and maintain that they do indeed supply the proofs of these laws in the classroom. However, the proof of the pudding lies in the tests and examinations. Was it made clear to the students that they would be required to demonstrate their knowledge and understanding of these proofs under examination conditions? If the answer is *no* then the proof was not included in the course. Students are
smart. They know when an instructor is merely talking and when the instructor is actually saying something.

Students come out of too many elementary course with a rote memorization of these important rules but with no idea how to derive them. Such students are doomed to a life of misery in the calculus sequence where they will be required to possess the problem solving skills that would have been imparted to them when they were studying the proofs of these laws.

Even worse, many elementary mathematics courses waste valuable time that could have been devoted to the study of these laws, and to the study of proper mathematical writing, by introducing natural logarithms and exponents. There is absolutely nothing to motivate natural logarithms and exponents in a precalculus course and, in particular, it is entirely inappropriate to include growth and decay problems at this level.

The Concept of a Limit

Suddenly, in the midst of all the slavish commitment to mechanical processes for churning out answers, many beginning calculus courses attempt to present a precise approach to limits. Suddenly, students who were not expected to work precisely with many elementary definitions and derivations of elementary facts are being expected to work with this, the hardest single concept in the entire mathematics course program. They go through a week or two of complete misery at the beginning of the calculus sequence until they are told, “April fool!! You don’t actually have to know all of this stuff. Now we’ll tell you what we shall really be doing in this course.”

And from that moment on, the calculus course degenerates into just one more maze of memorized formulas.

As a matter of fact, even the precise approach to limits is presented as a mechanical process. I once saw a multiple choice question in the AP calculus examination (yes, believe it or not, a multiple choice question on the precise approach to limits) that said:

\[ f(x) = 3x - 2, \ a = 1, \ \varepsilon = 4, \ \delta = ? \]

Then, unbelievably, the next question said:

\[ f(x) = 5x - 2, \ a = 3, \ \delta = 1, \ \varepsilon = ? \]

We have to laugh because otherwise we must weep. I only hope and pray that the idiot who made up that exam was never my student. I couldn’t stand the embarrassment.

The fact is that the moment at which the definition of a limit is properly understood is not the moment at which the student has picked up the \( \varepsilon-\delta \) ritual. The moment of understanding comes when one understands how the definition teams up with the completeness property of the real number system, to ensure that a function that is continuous on a closed interval must have a maximum and a minimum. This deep fact about continuous functions is beyond the scope of an elementary calculus sequence but it is the key ingredient in Rolle’s theorem which is the bedrock of differential calculus. Students emerge from a well taught first calculus course understanding the enormous importance of Rolle’s theorem and understanding that a future course will supply the missing step by showing why a continuous function on a closed interval must have a maximum and a minimum. They will be anxious to see that result when it is presented in a first course in real analysis.
Rules for Working Out Derivatives

A properly presented introduction to differential calculus will present a careful balance between techniques for working out derivatives and an understanding of the principles and students will depend on the quality of that balance for their success in their more advanced courses.

All too often, the power rule

$$\frac{d}{dx} x^p = px^{p-1}$$

is just memorized. Any instructor who risks demanding that the students should be able to write the details of

$$\frac{d}{dx} x^{3/5} = \lim_{t \to x} \frac{t^{3/5} - x^{3/5}}{t - x}$$

$$= \lim_{t \to x} \frac{(t^{1/5})^3 - (x^{1/5})^3}{(t^{1/5})^5 - (x^{1/5})^5}$$

$$= \lim_{t \to x} \frac{(t^{1/5} - x^{1/5})(t^{2/5} + t^{1/5}x^{1/5} + x^{2/5})}{(t^{4/5} + t^{3/5}x^{1/5} + t^{2/5}x^{2/5} + t^{1/5}x^{3/5} + x^{4/5})}$$

$$= \lim_{t \to x} \frac{t^{2/5} + t^{1/5}x^{1/5} + x^{2/5}}{t^{4/5} + t^{3/5}x^{1/5} + t^{2/5}x^{2/5} + t^{1/5}x^{3/5} + x^{4/5}}$$

$$= \frac{3}{5} x^{-2/5}$$

runs the risk of seeing lower student evaluations than the instructor who simply tells the students to memorize the power rule.

An instructor who ensures that students can write out the proof that every differentiable function is continuous and then apply this fact to the proof of the product rule for differentiation by saying that

$$\frac{d}{dx} f(x)g(x) = \lim_{t \to x} \frac{f(t)g(t) - f(x)g(x)}{t - x}$$

$$= \lim_{t \to x} \frac{f(t)g(t) - f(x)g(t) + f(x)g(t) - f(x)g(x)}{t - x}$$

$$= \lim_{t \to x} \left( f(t) - f(x) \right) g(t) + f(x) \left( g(t) - g(x) \right) \left( \frac{t - x}{t - x} \right)$$

$$= f'(x)g(x) + f(x)g'(x)$$

runs the risk of seeing lower student evaluations than the instructor who simply tells the students to memorize the product rule. Ditto the quotient rule.

An instructor of Calculus I who actually makes it his/her business to teach Calculus I runs the risk of seeing lower evaluations than the instructor who allows the course to degenerate into an anthology of memorized rules and who then encourages the students to waste their time on appalling software trash like My MathLab.

Calculus of the Trigonometric Functions

Hardly a student who emerges from a typical Calculus I course is unable to quote the fact
that
\[
\frac{d}{dx} \sin x = \cos x
\]
and
\[
\frac{d}{dx} \cos x = -\sin x
\]
and many of them are able to apply these equations to a variety of problems on curve sketching, maxima and minima and so on. However, my experience is that only very rarely does a student emerge from Calculus I with even the remotest idea of why these important equations are true.

More often than not, the instructor devotes several minutes to the explanation of the inequality
\[
\cos \theta < \frac{\sin \theta}{\theta} < 1
\]
for \(0 < \theta < \frac{\pi}{2}\) that is deduced from the following figure by comparison of the areas of two triangles and a circular sector.

Then this inequality is combined with the sandwich rule to show that
\[
\lim_{\theta \to 0} \frac{\sin \theta}{\theta} = 1
\]
and that
\[
\lim_{\theta \to 0} \frac{1 - \cos \theta}{\theta} = 0.
\]
The students sit patiently through this knowing that the instructor is merely talking, rather than actually saying something, because it is well known to them that this material will not appear in their tests and examinations. So they never learn it. What a shame! The derivation of the inequality
\[
\cos \theta < \frac{\sin \theta}{\theta} < 1
\]
is the only time in their entire undergraduate studies at which the importance of radian measure of an angle is revealed. The heart of the calculus of the trigonometric functions has been lost to them.

Typically, the course continues with a list of exercises on the trigonometric limits. The students come to attention. Here, at last is something that actually “belongs” to the course, meaning that they will be responsible for it. So they dutifully work through a bunch of
exercises making sure that, in each case, they can match their “answers” with those given in the back of the textbook.

The next item on the agendum is the application of these trigonometric limits to the derivation of the identity

\[
\frac{d}{dx} \sin x = \cos x
\]

\[
\frac{d}{dx} \sin x = \lim_{h \to 0} \frac{\sin(x + h) - \sin x}{h}
\]

\[
= \lim_{h \to 0} \frac{\sin x \cos h + \sin h \cos x - \sin x}{h}
\]

\[
= \lim_{h \to 0} \left( \frac{\sin h}{h} \cos x - \frac{1 - \cos h}{h} \sin x \right)
\]

\[
= \cos x - 0 \sin x
\]

The students are asleep again. No matter. They will come back to life when asked to work out \( \frac{d}{dx} x^2 \sin x \) in the next block of exercises. In this type of course, the entire deductive flavour of mathematics has been lost and the students have learned little more than to apply a sequence of rote memorized working steps. However, any instructor who attempts to teach Calculus I instead of teaching in this mechanical way risks lower student evaluations.

**The Role of Proper Testing in a Mathematics Course**

I believe that I have already made my point that the way in which students approach their studies and the kind of study that they will undertake is strongly dependent on the way that the material will appear in the tests and examinations. The testing process defines the course for them. Our students are busy people and they have no time to engage in activity that they do not see as an actual integral part of their course. They look to us for guidance to help them set their sights and to determine what constitutes proper study of the material. The mere fact that certain material may have been presented in the classroom does not guarantee that the students will recognize it as belonging to the course. Instead, they will see it as material to be studied, as material that belongs to the course if and only if will be represented meaningfully in the examination process.

I believe, therefore, that the nature and quality of student understanding of the contents of any mathematics course is as dependent on the design of the examination process as it is on the design of the course itself. An instructor may present a course with all the best intentions in the world but, if that course is examined only for mechanical churning out of “right answers” rather than for a careful balance that includes understanding of the mathematics, then the examination process has committed an act of sabotage against the course. This is why I have spoken out so strongly against “answer in a box” and multiple choice examination. Multiple choice examination is, indeed, an act of sabotage against everything we stand for. There is no justification for it, ever. While we, as a department, even possess a machine for reading those dirty little multiple choice cards, we should not be saying a word about teaching effectiveness.
The Role of a Proper Commitment to Correct Mathematical Writing

I have left the most difficult issue for last. To begin, I must explain that I have a rather strange viewpoint when it comes to the use of language. I believe that the quality of language that we use in mathematics needs to be at least as good as the quality of language that we use when checking out at Kroger. That principle will come as a shock to many students and even to some instructors.

I believe that it is incumbent both on instructors and on students to present what they are saying in clear, unambiguous, and meaningful language and to do so one hundred percent of the time. As I have said, every time we say one thing while meaning something else, we are imposing two tasks on our students, to translate what we have said into what we actually mean, and then to get on with the mathematics itself. In short, I believe that it is much harder to talk nonsense than to make sense and that a major ingredient of proper and effective mathematics teaching is presentation in clear and meaningful language.

Pretty much the same rules of grammar that apply to an ordinary English sentence also apply to a mathematical sentence and the words that we use belong to parts of speech that are almost identical to those that are used in an ordinary English sentence. In mathematical language we have a few extra parts of speech such as the binary operator. This is not the place to attempt to provide an exhaustive and definitive analysis of part of speech and I am certainly not delving into the world of mathematical logic. I am listing a few of them because their misuse is responsible for a great deal of meaningless language in our community and is making mathematics harder to study than it should be. I shall confine my attention to the following common parts of speech.

- **Nouns:** These include numbers, points in space, functions, and many other mathematical objects.
- **Verbs:** These are the binary relations that we use in mathematics. So, for example, "is parallel to" is a verb.

A good mathematical word processor like Scientific Workplace will understand verbs and space around them in a special way. So, for example, if $L_1$ and $L_2$ are line segments, the assertion that $L_1$ is parallel to $L_2$ can be expressed as $L_1 \parallel L_2$ using the binary relation symbol $\parallel$ that is found in the list of binary relations.

- **Conjunctions:** Conjunctions such as "and" and "or" play a fundamental role in mathematical sentences. Since a sentence of the type "If $P$ then $Q$, " is identical to the sentence "Either $\neg P$ or $Q$, " I have lumped conditional words like if with the conjunctions.
- **Adjectives:** For example, "positive" is an adjective.
- **Prepositions:** More often than not, prepositions are part of the description of a binary
operation (verb) or they are part of the wording of an adjective.

- **Unary and Binary Operators**: I’m not sure whether we have binary operators such as + and − in ordinary English sentences but, of course, they play a big role for us in mathematics.

- **Words that Refer to what We Have Said**: I’m thinking of words like “therefore”.

Some of the most damaging misuse of language is caused by confusion about the part of speech to which a word belongs in a mathematical sentence. In the paragraphs below, I shall list some common errors that I think have an adverse effect on mathematical teaching and learning. I could write volumes on this subject and am, with great difficulty, restraining myself from writing more.

**Misuse of Conjunctions**

1. Many of my students use the conjunction or in a strange way. They may say that $x = \frac{4}{6}$ or $\frac{2}{3}$. This misuse of or is so common that I feel sure that they must be picking it up from their teachers. Perhaps we need to provide high school teachers with the message that the correct symbol to be placed between $\frac{4}{6}$ and $\frac{2}{3}$ is the equal sign =.

2. When asked to solve the equation $x^2 + x - 6 = 0$, many students write something like the following:

$$x^2 + x - 6 = 0$$

$$(x - 2)(x + 3)$$

$$\frac{2}{3} - 3$$

I can only assume that this sort of thing is generated by years of inattention in the classroom to proper writing and is possibly the result of “answer in a box” or multiple choice testing.

**Misuse of Adjectives**

Mathematical writing is replete with false adjectives that serve only to muddy the water. I strongly believe that our students would learn more and would have an easier time learning mathematics if we could use adjectives in a reliable way.

1. **Parallel** is not an adjective. We should not say that line segments are parallel. Would it really be so difficult to say that two given line segments are parallel *to one another*? The problem here is that we have a binary relation masquerading as an adjective.

2. Although the word **injective** is a perfectly valid adjective describing a special property of a function, that dreaded word **surjective** is a fake adjective. It is also a fake word because, for all its scholarly looking Latin origin, it has no actual Latin origin. Well, ignoring the etymological objections to **surjective**, I could ask what it means to say that a given function $f$ is **surjective**. I gather that it means that the function is **onto**. Really? Onto what? The fact is that no sentence can be ended properly with a preposition. Saying that a given function is onto makes about as much sense as saying that I am sitting **on**. Of course, every function is onto its own range, so perhaps every function is **surjective**.
3. The word *negative* is an adjective and it is nothing else. The word negative, as it applies to real numbers, means *less than zero*. So it is perfectly true to say that the number $-4$ is negative. Now just as *hot water* means water that is hot, a *negative four* should mean a four that is negative. As we know, there is no such number. There is only one number $4$ in arithmetic and this number is positive. So, what is this “negative four” to which so many keep referring? Perhaps it’s like the Indian rope trick. Everyone has heard about it but no one has ever seen it.

This unfortunate problem with the word *negative* is actually the result of an attempt to use the word as an adjective masquerading as a unary operator.

I find it very unfortunate that, although we usually write $\frac{1}{x}$ for the multiplicative inverse of a (nonzero) number $x$, and call it *one over* $x$, we seem to give a very different treatment to the additive inverse. Why don’t we write the additive inverse of a number $x$ as $0 - x$???

Wouldn’t it be easier for the students if we treated these two very similar concepts in the same way???

And, if we later dropped the symbol $0$ from $0 - x$ we would still be reading $-x$ as *minus* $x$ instead of the nonsensical *negative* $x$.

**Confusion of the Words “Implies” and “Therefore”**

I strictly forbid my students from using the sign $\Rightarrow$ or the word Implies, except in a heading. It is perfectly acceptable for a heading to say something like

**Proof that Condition 1 Implies Condition 2**

but, except in this sort of context, they may not use *implies* when writing mathematics for me to view.

I have found that, pretty much 100% of the time, students who are using the $\Rightarrow$ sign use it incorrectly. More often than not, they are using it when they really mean *therefore*. I am sorry to have to say that, quite often, instructors are making the same error.

Well written mathematics uses short sentences and avoids “if...........then” statements unless the argument will be quite short, and so it also avoids even correct use of the word *implies*. Instead of making an assumption or describing a case with the word *if*, one can use “*we assume that....*” or “*suppose that...*”. The short sentences that are created when mathematics written in this kind of way are always easier for students to follow.

**Summing Up**

In this document, I have tried to lay out my ideas about teaching effectiveness and what we can do to enhance teaching effectiveness, particularly in mathematics. The subject is vast and complicated and it is my earnest hope that the academic community will stop dumbing it down into a process of seeking the “right questions” to ask on an anonymous student evaluation form.

Again, I appeal to the academic community to care about teaching effectiveness rather than just using it as a yardstick.