Making Sense of Multiplying and Dividing Decimal Numbers: Problems, Representations, and Reasoning

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M4N5. Students will further develop their understanding of the meaning of decimals and use them in computations.

d. Model multiplication and division of decimals by whole numbers.
e. Multiply and divide both one and two digit decimals by whole numbers.
M5N3. Students will further develop their understanding of the meaning of multiplication and division with decimals and use them.

a. Model multiplication and division of decimals.
b. Explain the process of multiplication and division, including situations in which the multiplier and divisor are both whole numbers and decimals.
c. Multiply and divide with decimals including decimals less than one and greater than one.
d. Understand the relationships and rules for multiplication and division of whole numbers also apply to decimals.
4.NF  
**Build fractions from unit fractions by applying and extending previous understandings of operations on whole numbers.**  
4. Apply and extend previous understandings of multiplication to multiply a fraction **by a whole number.**

5.NBT  
**Perform operations with multi-digit whole numbers and with decimals to hundredths.**  
7. Add, subtract, **multiply**, and **divide** decimals to hundredths, using concrete models or drawings and strategies based on place value, properties of operations, and/or the relationship between addition and subtraction; relate the strategy to a written method and explain the reasoning used.
5.NF
Apply and extend previous understandings of multiplication and division to multiply and divide fractions.

4. Apply and extend previous understandings of multiplication to multiply a fraction or whole number by a fraction.

7. Apply and extend previous understandings of division to divide unit fractions by whole numbers and whole numbers by unit fractions.
5.NF  Footnote on Standard 7
Students able to multiply fractions in general can develop strategies to divide fractions in general, by reasoning about the relationship between multiplication and division. But division of a fraction by a fraction is not a requirement at this grade.
6.NS

Apply and extend previous understandings of multiplication and division to divide fractions by fractions.

1. Interpret and compute quotients of fractions, and solve word problems involving division of fractions by fractions, e.g., by using visual fraction models and equations to represent the problem.
So, why do GPS and CCSS distinguish multiplying and dividing “by whole numbers” and “by decimal numbers (or fractions)”?
3.OA
Represent and solve problems involving multiplication and division.

1. Interpret products of whole numbers, e.g., interpret $5 \times 7$ as the total number of objects in 5 groups of 7 objects each.

$\text{(Number of Groups)} \times \text{(Number in a Group)} = \text{(Total Number)}$

“(Number in a Group) is multiplied by (Number of Groups).”

Convention on Notations (CCSS)
5 x 7 = 15

“5 times 7 equal 15.”
“7 multiplied by 5 equal 15.”
(Number of Groups) is a whole number.

Example:
7 x 3.45 – 7 groups of 3.45, or “3.45 multiplied by 7”

Non-example:
3.45 x 7 – 3.45 groups (???) of 7, or “7 multiplied by 3.45”

“by whole number”
Example

- Jenny bought 7 boxes of cereal. If each box contains 3.45 pounds of cereal, what is the total weight of cereal she bought?

Non-example

- One liter of paint weighs 7 pounds. How much will 3.45 liters of the same paint weigh?
Example

- Jenny bought 7 boxes of cereal. If each box contains 3.45 pounds of cereal, what is the total weight of cereal she bought?

\[ 7 \times 3.45 = ? \]
\[ ? = 3.45 + 3.45 + 3.45 + 3.45 + 3.45 + 3.45 + 3.45 \]

Repeated addition makes sense
3.45 x 7 = ?

? = is somewhere in between 7+7+7 and 7+7+7+7

Non-example
- One liter of paint weighs 7 pounds. How much will 3.45 liters of the same paint weigh?

Repeated addition cannot be applied
But isn’t

\[3.45 \times 7 = 3.45 + 3.45 + \ldots\]

But we don’t have 7 groups of 3.45 liters!

Non-example

- One liter of paint weighs 7 pounds. How much will 3.45 liters of the same paint weigh?

Example & Non-example
multiply by a whole number:
The equal group meaning of multiplication is applicable.

multiply by a non-whole number:
The meaning of multiplication needs to be expanded.
CCSS 5.NF

5. Interpret multiplication as scaling (resizing)

3.45 x 7: 3.45 times as much as 7
M3N1. Students will further develop their understanding of whole numbers and decimals and ways of representing them.

b. Understand the relative sizes of digits in place value notation (10 times, 100 times, 1/10 of a single digit whole number) and ways to represent them including word name, standard form, and expanded form.
2734:

2000 + 700 + 30 + 4
2 \times 10^3 + 7 \times 10^2 + 3 \times 10 + 4 \times 1

0.45:

0.4 + 0.05
4 \times 0.1 + 5 \times 0.01
2734: How many 100s does this number have? How many 10s does this number have?

2734: 27 100s and 34 273 10s and 4
0.45: 45 0.01s
Fractions, too

\(\frac{3}{5}: 3 \ 1/5s\)
All of these are $3 + 4$:

- $30 + 40$
- $30000 + 40000$
- $0.3 + 0.4$
- $0.003 + 0.004$
- $3/9 + 4/9$
4 \times 2.7 = ?

“How much is 4 groups of 2.7?”

2.7 is the same as 27 0.1s.

“How much is 4 groups of 27 0.1s?”

4 \times 27 = 108 0.1s.

4 \times 2.7 = 10.8

By whole numbers
- \(6 \times 3.8 = 22.8\)
- \(7 \times 2.14 = 14.98\)
- \(12 \times 7.03 = 84.36\)
- ...

- The units for the multiplicand and the product are the same.
7.2 ÷ 4 = ?

If you make 4 equal groups with 7.2, how much is in each group?

If you make 4 equal groups with 72 0.1s, how much is in each group?

\[ 72 ÷ 4 = 18 \text{ 0.1s} \]

\[ 7.2 ÷ 4 = 1.8 \]

By whole numbers
Let’s practice

- $8.4 \div 3 = ?$
- $6.15 \div 5 = ?$
7.2 ÷ 4 = ?
If you make 4 equal groups with 7.2, how much is in each group?

If you make 4 equal groups with 72 0.1s, how much is in each group?
72 ÷ 4 = 18 0.1s

7.2 ÷ 4 = 1.8

By whole numbers
What about \(7.4 \div 4 = ?\)

If we make 4 equal groups with 7.4, how much is in each group?

If we make 4 equal groups with 74 0.1s, how much is in each group?

\[
74 \div 4 = 18 \text{ 0.1s} \quad \text{... rem.} \quad 2 \text{ 0.1s}
\]

\[
7.4 \div 4 = 1.8 \quad \text{... rem} \quad 0.2
\]
74 ÷ 4 = 18 0.1s ... rem. 2 0.1s

2 0.1s → 20 0.01s
Make 4 equal groups with 20 0.01s.
Each will get 5 0.01s.

7.4 ÷ 4 = 1.85
A 1 m wire weighs 2.8 g. How much will 3.6 m of the same wire weigh?

\[ 3.6 \times 2.8 = ? \]

“3.6 times as much as 2.8”

By decimal numbers
3.6 \times 2.8 = ?

3.6 \, m \rightarrow 36 \, 0.1 \, m

By decimal numbers
3.6 \times 2.8 = ?

3.6 \, m \rightarrow 36 \, 0.1 \, m

If we know how much a 0.1m wire weighs, we can figure out the total.

Wt. of 0.1m = Wt. of 1m \div 10 = 0.28

36 \times 0.28 =

By decimal numbers
3.6 \times 2.8 = ?

3.6 \, m \rightarrow 36 \, 0.1 \, m

If we know how much a 0.1m wire weighs, we can figure out the total.

\text{Wt. of 0.1m} = \frac{\text{Wt. of 1m}}{10} = 0.28

36 \times 0.28 = 36 \times (28 \, 0.01s)

\textbf{By decimal numbers}
3.6 \times 2.8 = ?

3.6 \, m \rightarrow 36 \, 0.1 \, m

If we know how much a 0.1m wire weighs, we can figure out the total.

\text{Wt. of 0.1m} = \text{Wt. of 1m} \div 10 = 0.28

36 \times 0.28 = (36 \times 28) \, 0.01s

\textbf{By decimal numbers}
3.6 \times 2.8 = ?

3.6 \, m \rightarrow 36 \, 0.1 \, m

If we know how much a 0.1\, m wire weighs, we can figure out the total.

Wt. of 0.1\, m = Wt. of 1\, m \div 10 = 0.28

36 \times 0.28 = 1008 \, 0.01s

By decimal numbers
3.6 \times 2.8 = ?

3.6 \, m \rightarrow 36 \, 0.1 \, m

If we know how much a 0.1m wire weighs, we can figure out the total.
Wt. of 0.1m = Wt. of 1m \div 10 = 0.28
36 \times 0.28 = 10.08

By decimal numbers
3.6 \times 2.8 = ?

**Unit Approach**
3.6 × 2.8 = ?

2.8 ÷ 10 = 0.28

amount per 0.1 of the multiplicand

36 × 0.28

number of 0.1-units in the multiplicand

36 groups of 0.28

36 groups of 28 0.01s

36 × 28 = 1008 0.01s

3.6 × 2.8 = 36 × 0.28 = 10.08

Unit Approach
3.6 \times 2.8 = ?

Unit Approach
Let’s practice!

4.3 \times 7.2 = ?

2.54 \times 1.8 = ?
A 1 \textit{m} wire weighs 2.8 \textit{g}. How much will 3.6 \textit{m} of the same wire weigh?
A 1 m wire weighs 2.8 g. How much will 3.6 m of the same wire weigh?

\[ 3.6 \times 2.8 = ? \]

What if we had 36 m of the wire? How much will it weigh?

\[ 36 \times 2.8 = 100.8 \text{ g} \]

36 m is 10 times as much as 3.6 m. 100.8 g is 10 times as much, too.

\[ 3.6 \times 2.8 = 100.8 \div 10 = 10.08 \]
3.6 \times 2.8

\[ \text{Make-It-Bigger} \]
Let’s practice!

4.3 \times 7.2 = {?}

2.54 \times 1.8 = {?}

Let’s practice!
Developing Algorithm

4.3 x 7.2 = ?

\[
\begin{align*}
4.3 \times 7.2 & \rightarrow 43 \\
43 & \div 10 - \\
\end{align*}
\]

2.54 x 1.8 = ?

\[
\begin{align*}
2.54 \times 1.8 & \rightarrow 254 \\
254 & \div 100 - \\
\end{align*}
\]
A 1.8 m wire weighs 4.5 g. How much will 1 m of the same wire weigh?

\[ \frac{4.5}{1.8} = ? \]

**Dividing by decimals**
A 1.8 m wire weighs 4.5 g. How much will 1 m of the same wire weigh?

\[ 4.5 \div 1.8 = ? \]
A 1.8 \text{ m} wire weighs 4.5 \text{ g}. How much will 1 \text{ m} of the same wire weigh?

\[ 4.5 \div 1.8 = ? \]

1.8 → 18 0.1s

Dividing by decimals
A 1.8 m wire weighs 4.5 g. How much will 1 m of the same wire weigh?

\[
4.5 \div 1.8 = ?
\]

1.8 → 18 0.1s

\[
4.5 \div 18 = 0.25: \text{ Weight of 0.1 m}
\]

0.25 x 10 = 2.5: Weight of 1 m

\[
4.5 \div 1.8 = 2.5
\]
A 1.8 m wire weighs 4.5 g. How much will 1 m of the same wire weigh?

\[ 4.5 \div 1.8 = ? \]

1.8 \(\rightarrow\) 18 0.1s

\[ 4.5 \div 18 = 0.25: \text{Weight of 0.1 m} \]

\[ 0.25 \times 10 = 2.5: \text{Weight of 1 m} \]

\[ 4.5 \div 1.8 = 2.5 \]
Unit Approach
A 1.8 m wire weighs 4.5 g. How much will 1 m of the same wire weigh?

What if we had 18 m of the wire? Weight of 18 m: 10 \times 4.5 = 45

The weight of 1 m did not change. If 18 m weighs 45 g, how much will 1 m weigh? 45 \div 18 = 2.5.
Make-It-Bigger
Let’s practice!

62.9 ÷ 3.7 = ?
8.4 ÷ 3.36 = ?
7.44 ÷ 6.2 = ?
62.9 ÷ 3.7 = ?
  What if we had 37?
  629 ÷ 37 = 17

8.4 ÷ 3.36 = ?
  What if we had 336?
  840 ÷ 336 = 2.5

7.44 ÷ 6.2 = ?
  What if we had 62?
  74.4 ÷ 62 = 1.2

Developing algorithm
Make-It-Bigger
With 1 dl of paint you can paint $\frac{3}{5} m^2$ of boards. How many $m^2$ can you paint with 2 dl of paint?
2 \times \frac{3}{5}

2 \text{ groups of } \frac{3}{5}, \text{ or }
2 \text{ groups of } 3 \frac{1}{5}\text{s}

2 \times 3 = 6 \frac{1}{5}\text{s}

2 \times \frac{3}{5} = \frac{6}{5}
In general,

\[ \frac{a}{b} \times c = \frac{a \times c}{b} \]
With $3\text{dl}$ of paint you can paint $\frac{4}{5} m^2$ of boards. How many $m^2$ can you paint with $1\text{dl}$ of paint?
4/5 ÷ 3
Make 3 equal groups with 4/5
4 1/5s can’t be made into 3 equal groups...
4x3/5x3=12/15, or 12 1/15s, can be made into 3 equal groups.
12 ÷ 3 = 4, 4 1/15s in each.
4/5 ÷ 3 = 4/15
In general,

\[
\frac{a}{b} \div c = \frac{a}{b \times c}
\]
1. With 1 dl of paint you can paint $\frac{4}{5}$ m$^2$ of boards. How many m$^2$ can you paint with $\frac{2}{3}$ dl of paint?
With 1 dl of paint you can paint \( \frac{4}{5} \) m\(^2\) of boards.

How many m\(^2\) can you paint with \(\frac{2}{3}\) dl of paint?

(amt. of paint) \(\times\) (area you can paint with 1 dl)

**Fraction Multiplication**
With 1 dl of paint you can paint \( \frac{4}{5} \) m\(^2\) of boards. How many m\(^2\) can you paint with \( \frac{2}{3} \) dl of paint?

\[
\frac{2}{3} \times \frac{4}{5} = ?
\]

Fraction Multiplication

How much can we paint with $1/3$ dl?

$4/5 \div 3 = 4/(5 \times 3) = 4/15$

$2/3$ dl is twice of $1/3$ dl. So, the area we can paint with $2/3$ dl will be $2 \times 4/15 = 8/15$
\[
\begin{array}{c@{\times}c@{=}c}
2 & 4 & 8 \\
3 & 5 & 15 \\
2/3 & 3 & 2
\end{array}
\]

How much area can we paint with 2 \textit{dl}?

\[
4/5 \times 2 = 8/5 \text{ m}^2
\]

But this is 3 times as much as what we can paint with 2/3 \textit{dl}.

Area we can paint with 2/3 \textit{dl}

\[
8/5 \div 3 = 8/(5 \times 3) = 8/15
\]
\[
\frac{2}{3} \times \frac{4}{5} = \frac{8}{15}
\]

Make-It-Bigger
With $\frac{3}{4} \text{dl}$ of paint you can paint $\frac{2}{5} \text{m}^2$ of board.

How many $\text{m}^2$ can you paint with $\text{1dl}$ of paint?

Hironaka & Sugiyama (2002), p. 17
1. With $\frac{3}{4}$ dl of paint you can paint $\frac{2}{5}$ m$^2$ of board. How many m$^2$ can you paint with 1 dl of paint?

\[
\frac{\text{Area painted}}{\text{Amt. of paint}}
\]

\textbf{Fraction Division}

Hironaka & Sugiyama (2002), p. 17
With $\frac{3}{4}$ dl of paint you can paint $\frac{2}{5}$ m² of board. How many m² can you paint with 1 dl of paint?

$$\frac{2}{5} \div \frac{3}{4} = ?$$

**Fraction Division**

Hironaka & Sugiyama (2002), p. 17
\[
\frac{2}{5} \div \frac{3}{4} = ?
\]

Unit Approach
\[
\frac{\frac{2}{5}}{\frac{3}{4}} = 4 \times \left(\frac{\frac{2}{5}}{3}\right) = \frac{2 \times 4}{5 \times 3} = \frac{2}{5} \times \frac{3}{4}
\]
CCSS treats multiplication and division of decimals in Grade 5...
... after students study multiplication of fractions by whole numbers.
In Grade 5, students learn multiplication by fractions, and
Dividing unit fractions by whole numbers, or dividing whole numbers by unit fractions.
In Grade 5, students are also to learn multiplication as scaling.

HOW?
• In Grade 5, students are also to learn multiplication as scaling.

• Through learning multiplication of decimals and fractions.
• In Grade 5, students are also to learn multiplication as scaling.

• Through learning multiplication of decimals and fractions.
Looking Ahead

- Multiplication as scaling
- Division to find the “base” number
- Division to find the scale factor
Looking Ahead

- Multiplication as scaling
- Division to find the “base” number
  - Fair sharing division – how many in each group?
- Division to find the scale factor
  - Measurement (repeated-subtraction) division – how many groups.

Looking Ahead

MATHEMATICS: CONNECTING THE PIECES
Multiplication as scaling

Division to find the “base” number
  ◦ Fair sharing division – how much per one unit?

Division to find the scale factor
  ◦ Measurement (repeated-subtraction) division – how many times as much?
The reason we teach multiplication and division of decimals and fractions isn’t just helping students know computation methods.

Problem solving and developing productive dispositions are also important.
• We should think about which (fractions of decimals) we should teach first and why.
• We should examine carefully how this topic is approached in the CCSS.