Challenges and Opportunities for Teaching and Learning of Proofs

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Pre-K – 12 curriculum should enable all students to:
- recognize reasoning and proof as fundamental aspects of mathematics;
- make and investigate mathematical conjectures;
- develop and evaluate mathematical arguments and proofs;
- select and use various types of reasoning and methods of proof.
Opportunities & Challenges

• Opportunities: “…proof is expected to play a much more prominent role through the entire school mathematics curriculum and to be a part of the mathematics education of all students.” (Knuth, 2002, p. 380)

• Challenges: “… proof is expected to play a much more prominent role through the entire school mathematics curriculum and to be a part of the mathematics education of all students.” (Knuth, 2002, p. 380)
Challenges

• Motivating students

Why do we need to prove it?
We already know it’s true!
In elementary schools,

2. Let’s compare the angles of an isosceles triangle and an equilateral triangle!

1. Please draw an isosceles triangle and cut it out. Then fold it like the picture below and compare the angles.

2. Let’s do the same thing with an equilateral triangle!
In elementary schools,

**Diagonals**

4. Draw straight lines in the quadrilaterals below by connecting opposite vertices.

- ![Quadrilaterals](image)

Line segments that connect two opposite vertices are called **diagonals**.

2. Let’s investigate the characteristics of diagonals in quadrilaterals!

1. Compare the lengths of the two diagonals.
2. Compare the lengths of the four line segments between the intersection of the diagonals and the four vertices.
3. Compare the angles formed by the two diagonals at the point of intersection.
In elementary schools,

4 Using these two friends’ methods, find the sum of the 3 angles of a triangle.

The sum of the 3 angles of a triangle is 180°.
Convincing argument?

• Students are already convinced.
• Their elementary school teachers told them.
• Examples and empirical evidences are convincing enough to many students.
Purposes of Proof

• verification
• explanation
• systematization
• discovery
• communication
• intellectual challenge

De Villiers, 1999
Conviction as Motivation

In actual mathematical practice, conviction is far more frequently a prerequisite for the finding of a proof than doubt.

(DeVilliers, 2004, p. 705)
Proof as Explanation

• Using two different digits from 1, 2, 3, ..., 9, make a 2-digit number. Reverse the order of digits to make a 2nd 2-digit number.

• Find their sum.

• The sum is always divisible by 11.
Empirical Verification

- $35 + 53 = 88$, and $11$ divides $88$.
- $82 + 28 = 110$, and $11$ divides $110$.
- $97 + 79 = 176$, and $11$ divides $176$.
- etc.
Proof

• Let $a$ and $b$ be the two digits.
• The two 2-digit numbers are: $10a + b$ and $10b + a$.
• Their sum, therefore, is $10a + b + 10b + a$.
• You can simplify this to $11a + 11b$.
• So, the sum is $11(a + b)$, a multiple of 11.
Proofs in Japanese Textbook

• Grade 8
• 1992
  – Parallel lines and angles
  – Sum of interior angles of a triangle
  – Interior and exterior angles of polygons
• 2006
  – Interior and exterior angles of polygons
  – Parallel lines and angles
  – Sum of interior angles of a triangle
A new approach to teaching proofs

• Instead of focusing on verification and (global) systematization, help students experience local systematization.
Local Systematization

Given (in elementary school)

• Sum of angles in a triangle is $180^\circ$.

New relationship.

• Sum of angles in a $n$-gon is $180^\circ \times (n-2)$. 
180 \times (\text{number of triangles})
Sum of angles = $180 \times n - 360$

$= 180 \times (n - 2)$
Local Systematization

Given (in elementary school)

• Sum of angles in a triangle is $180^\circ$.

New relationship.

• Sum of angles in a $n$-gon is $180^\circ \times (n-2)$.

Which leads to,

• Sum of external angles of a $n$-gon is $360^\circ$.
exterior angle + interior angle = 180°

exterior angle = 180° - interior angle

sum of exterior angles = 180° \times 5 + \text{sum of angles} 

= 900° - 540° 

= 360°
Sum of Exterior Angles in a *n*-gon

\[
\text{sum of exterior angles} = 180^\circ \times n - 180^\circ (n - 2)
\]

\[
= 180^\circ \times 2
\]

\[
= 360^\circ
\]
Local Systematization

• Sum of angles in an $n$-gon is $180^\circ \times (n - 2)$ **BECAUSE** sum of angles in a triangle is $180^\circ$.

• Sum of exterior angles in an $n$-gon is $360^\circ$ **BECAUSE** sum of interior angles in an $n$-gon is $180^\circ \times (n - 2)$.

• So, what is the reason that sum of angles in a triangle is $180^\circ$?
Challenges

• Difficulty with techniques with proving – auxiliary lines
Some of the principal auxiliary lines used on rectilinear figures are:

1. A line connecting two given points.
2. A line through a given point parallel to a given line.
3. A line through a given point perpendicular to a given line.
4. A line making a given angle with a given line.
In demonstrating theorems relating to the circle, it is often helpful to draw one or more of the following auxiliary lines: A radius, a diameter, a chord, a perpendicular from the center upon a chord, an arc, a circle, etc. (p. 150)
問10 右の図で \( \ell \parallel m \) であるとき、
\[ \angle x \] の大きさを求めなさい。

問1 右の図で
\[ \angle ADC = \angle A + \angle B + \angle C \]
が成り立つ。このことをいろ
いろな方法で証明してみよう。
If $l \parallel m$, find the measure of angle $x$. 
Possible Auxiliary Lines

\[ x = 60 + 40 = 100^\circ \]
In this diagram, $\angle ADC = \angle A + \angle B + \angle C$. Prove this in many different ways.
Using Parallel Lines
Using Parallel Lines
Using Parallel Lines
Using Exterior/Interior Angles
Using Exterior/Interior Angles
Using Exterior/Interior Angles
Challenges

• “…proof is expected to play a much more prominent role through the entire school mathematics curriculum and to be a part of the mathematics education of all students.” (Knuth, 2002, p. 380)
Reasoning & Proof in pre-K - 12

• Need a more developmental view of “proof” – that is, what is “proof” in elementary schools?

• Can such a view of proof be consistent with proof in HS geometry (and beyond)?
Definition of Proof
(Stylianides, 2007, p. 291)

Proof is a mathematical argument, a connected sequence of assertions for or against a mathematical claim, with the following characteristics:

1. It uses statements accepted by the classroom community (set of accepted statements) that are true and available without further justification;
Definition of Proof (Stylianides, 2007, p. 291)

Proof is a mathematical argument, a connected sequence of assertions for or against a mathematical claim, with the following characteristics:

2. It employs forms of reasoning (modes of argumentation) that are valid and known to, or within the conceptual reach of, the classroom community;
Definition of Proof (Stylianides, 2007, p. 291)

Proof is a mathematical argument, a connected sequence of assertions for or against a mathematical claim, with the following characteristics:

3. It is communicated with forms of expression (modes of argument representation) that are appropriate and known to, or within the conceptual reach of, the classroom community.
3 Dimensions of Development

- Set of mathematical concepts & procedures
- Repertoire of argumentation
- Set of representations
The unit digit of $5^n$ for all natural number $n$ is 5.

- Elementary school
  
  \[ 5^2 = 25; \ 5^3 = 125; \ 5^4 = 625; \ 5^5 = 3125, \ldots \]

- Grade 5 classroom (Lampert, 1990)
  
  “You don’t have to do that (actually multiply 25x25). It’s easy, the last digit is always going to be 5 because you are always multiplying last digits of 5, and 5 times 5 ends in a 5.” (p. 48)
The unit digit of $5^n$ for all natural number $n$ is 5.

• Through mathematical induction
  
  (1) Prove it is true when $n = 1$: $5^1 = 5$, $\rightarrow$ true.
  
  (2) Assume it is true when $n = k$.
  
  (3) Prove it is true for $n = k + 1$
      
      $5^{k+1} = 5^k \times 5$
      
      The unit digit for $5^k$ is 5.
      
      Therefore, the unit digit of $5^{k+1}$ is the unit digit of $5 \times 5$, which is a 5.
      
  (4) (1), (2), and (3) $\rightarrow$ the statement is true.
Generic Argument

• Suppose we have 94 and 49.
• If you write the vertically, we have:
  \[
  \begin{array}{c}
  94 \\
  +49 \\
  \end{array}
  \]
• We can switch the ones digits without changing the sum.
• So, we have 99 + 44, and 11 divides both.
• So, the sum must be divisible by 11, too.
Principles and Standards for School Mathematics (NCTM, 2000)

• From children's earliest experiences with mathematics, it is important to help them understand that assertions should always have reasons. (p. 56)

• By the upper elementary grades, justifications should be more general and can draw on other mathematical results. (p. 58)
Principles and Standards for School Mathematics (NCTM, 2000)

- Students should move from considering individual mathematical objects -- this triangle, this number, this data point -- to thinking about classes of objects -- all triangles, all numbers that are multiples of 4, a whole set of data. (p. 188)
Using a Definition

Which ones are isosceles or equilateral triangles in the picture below?

How can I find them?
Using a Property

3. Draw the diameters for the circles that share the same center as shown on the left. If we connect the end points of diameters, A, B, C, and D, what kind of quadrilateral will we get?
In Grade 8

• Chapter 1: Literal expressions/equations

Investigate the characteristics of the sums of 5 consecutive whole numbers by examining several examples.

Explain using the literal expressions why the sum of 5 consecutive whole numbers is a multiple of 5.
In Grade 8

• Chapter 3: Systems of Equations

3种類のおもり○, △, □があります。下の図1・図2のようにおもりをてんぴんの左右のさらにのせると，てんぴんはつり合います。図3のように右のさらに□のおもりを2個のせたとき，てんぴんをつり合わせるには，アのさらに△のおもりを何個のせればよいですか。△のおもりの個数を求めなさい。また，そう考えた理由を，式を使って説明しなさい。
• “…proof is expected to play a much more prominent role through the entire school mathematics curriculum and to be a part of the mathematics education of all students.” (Knuth, 2002, p. 380)

• “This abrupt transition to proof is a source of difficulty for many students, even for those who have done superior work with ease in their lower-level mathematics courses” (Moore, 1994, p. 249).
References


Thank you!